The superfluid phases of quark matter: Ginzburg-Landau theory and color neutrality

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(February 1, 2008)

We systematically apply Ginzburg-Landau theory to determine BCS pairing in a strongly-coupled uniform superfluid of three-flavor massless quarks in flavor equilibrium. We elucidate the phase diagram near the critical temperature in the space of the parameters characterizing the thermodynamic-potential terms of fourth order in the pairing gap. Within the color and flavor antisymmetric channel with zero total angular momentum, the phase diagram contains an isoscalar, color-antitriplet phase and a color-flavor-locked phase, reached by a second order phase transition from the normal state, as well as states reached by a first order phase transition. We complement the general Ginzburg-Landau approach by deriving the high-density asymptotic form of the Ginzburg-Landau free energy from the finite temperature weak-coupling gap equation. The dynamically-screened, long-range color magnetic interactions are explicitly taken into account in solving the gap equation. We find that in the limit of weak coupling, the isoscalar, color-antitriplet phase has higher free energy near the transition temperature than the color-flavor locked phase.

In view of the fact that deconfined quark matter must be color charge neutral, we incorporate the constraint of overall color neutrality into the general Ginzburg-Landau theory and the gap equation. This constraint yields a disparity in the chemical potential between colors and reduces the size of the pairing gap, in the presence of the anisotropy of the order parameters in color space. In comparison with the case in which there are no chemical potential differences between colors and hence the superfluid generally has nonzero net color charge, we find that while the constraint of color neutrality has only negligible effects on the gap in the weak coupling regime, it appreciably destabilizes the isoscalar, color-antitriplet phase in the strong coupling regime without affecting the color-flavor-locked phase.

I. INTRODUCTION

The possibility that degenerate relativistic quark matter becomes a color superconductor at low temperatures has been considered for the past two decades. As the seminal papers [1,2] noted, the quark-quark interaction in the color antitriplet channel is attractive and drives a Cooper pairing instability in the system, even in the limit of high densities where the Fermi energy of the quarks dominates over the one-gluon exchange interaction energy. Work in the intervening period has concentrated on the mean field approach, strictly valid only in the weak coupling limit (see, e.g., Refs. [3] and [4] for reviews). As the density is lowered, nonperturbative effects arising from self-couplings of the gluon field prevail, finally leading to a confinement transition into hadronic matter. Strong coupling effects can modify the equilibrium order parameter from the weak coupling prediction, as effects of spin fluctuation exchange do in superfluid ^{3}He [5]. The resultant change in the color-superconducting phase can also affect the breaking of chiral symmetry [6–8]. The properties of the color superconductivity such as the pairing gap and the critical temperature, T_c, have yet to be derived in the strong coupling regime in a rigorous way.

In this paper we examine color superconductors by means of a general Ginzburg-Landau approach, which permits us to determine the coarse grained features of such systems at temperatures just below T_c, for arbitrary QCD coupling constant, g. This approach, pioneered by Anderson and Brinkman [9] and Mermin and Stare [10] in the context of superfluid ^{3}He, reveals the most energetically favorable phase just below T_c, in terms of the parameters characterizing the thermodynamic potential to fourth order in the pairing gap. The first application of the general Ginzburg-Landau theory was made by Bailin and Love [2] to quark pairing with one flavor and total angular momentum $J = 1$. Here we shall consider more general Cooper pairing between u, d, and s quarks with $J = 0$. We complement this approach by deriving the parameters controlling the fourth order terms from the weak coupling gap equation. Previous work [11,12] has systematically investigated such pairing at zero temperature; the present study provides a systematic elucidation of the equilibrium properties near T_c.

The zero-temperature pairing gap and hence T_c is predicted to be $\sim$ 10–100 MeV for baryon chemical potentials $\sim$ 1 GeV [13]. This prediction relies on extrapolation from weak coupling into the low density, nonperturbative regime, by incorporating into the BCS gap equations effective interactions modeled after instanton-mediated interactions, in such a way as to reproduce constituent quark masses. Application of the Wilson renormalization group to analysis of the stability of a Fermi liquid against Cooper pairing [14] suggests that four-fermion couplings, induced between two particles with zero total momentum by one-gluon exchange or by instantons, grow logarithmically with momentum as higher modes are successively integrated out closer to the Fermi surface; the scattering amplitudes eventually reach a singularity – a Landau pole – in a way dependent on the number of flavors involved in the pairing. For two flavors (u, d), both gluon and instanton-induced interactions play a role in opening an energy gap in the isoscalar, color-antitriplet channel with zero total angular momentum $(J = 0)$. Pairing in this channel partially breaks baryon number symmetry and global color rotational invariance, but restores chiral symmetry. For three flavors (u, d, s), on the other hand, a color-flavor locked state [15] arises in the $J = 0$ channel mainly from the gluon-induced interactions when the strange quark mass is sufficiently small. This state is invariant under simultaneous exchange of color and flavor, but not under single exchange of color or flavor; chiral symmetry as well as baryon number symmetry and global color rotational invariance are broken. In the high density regime where the interactions are dominated by one-gluon exchange, the color magnetic (transverse) force, which is screened only dynamically by Landau damping of its mediators [16,17], is sufficiently long ranged to alter the dependence of the pairing gap on the QCD coupling constant g from the BCS result. This fact was first noted by Son [18] using a renormalization group method and an approximate solution to the relevant gap equation.

As in prior papers, we focus on the equilibrium properties of an ultrarelativistic color superconductor that is *homogeneous*, in the sense that the real gluon field vanishes everywhere and the order parameter is everywhere continuous in magnitude and orientation; we denote such a state as *superfluid quark matter*. Such homogeneity is similar to that in superfluid ^{3}He and superfluid neutron matter, as noted by Bailin and Love [2], because in both cases breaking of the global $U(1)$ gauge symmetry is accompanied by global symmetry breaking associated with the internal degrees of freedom. In superfluid quark matter, the possible order parameters are generally *anisotropic* in color space (see, e.g., Ref. [2]), a situation analogous to superfluid ^{3}He in which, as seen experimentally, the anisotropy lies in spin space [5].

We first restrict ourselves to the case, normally assumed in earlier investigations, in which the chemical potentials of different color and flavor quarks are equal. We obtain the thermodynamic potential difference between the superfluid and normal phases near T_c from the Ginzburg-Landau approach. The terms of second and fourth order in the pairing gap are constrained by invariances of the grand canonical Hamiltonian and by the structure of the order parameters, assumed here to be antisymmetric in color and flavor space. We then identify the degenerate sets of order parameters corresponding to local energy minima as isoscalar color-antitriplet and color-flavor locked states, and determine their condensation energies. We draw the resultant equilibrium phase diagram near T_c in the space of parameters characterizing the fourth order terms. We find that in the limit of weak coupling, the isoscalar, color-antitriplet phase is less favorable than the color-flavor locked phase near T_c.

In general, determination of the Ginzburg-Landau parameters requires inclusion of strong coupling effects. In the weak coupling limit, we anchor the general Ginzburg-Landau approach by deriving the parameters from the relevant weak coupling gap equations at finite temperature [2,19,20], including the infrared structure of the gluon propagator as in Ref. [20]. The latter behavior determines the g dependence of the pairing gap in the weak coupling limit. Debye screening of the electric gluons and Landau damping of the magnetic gluons, calculated with one-gluon exchange as modified by a normal medium, provide effective infrared cutoffs of the respective scattering amplitudes.

Motivated by the fact that deconfined quark matter must be in an overall color singlet state, we also consider the equilibrium properties of superfluid quark matter under the condition that it is color-charge neutral. In normal quark matter, in which global color rotational invariance is ensured by QCD interactions, the constraint of color neutrality leads to equality of the chemical potentials for quarks of different colors (see, e.g., Ref. [21]). Superfluid quark matter, on the other hand, generally has a preferred direction in color space. Such violation of color rotational invariance together with the requirement of overall color neutrality leads to differences in the chemical potential between colors, in a way dependent on the degree of anisotropy of the order parameters in color space. This disparity in chemical potentials in turn alters the pairing gap, a feature seen in the general Ginzburg-Landau analysis. Beyond the usual terms of second and fourth order in the pairing gap, color neutrality adds terms dependent on the chemical potential differences between colors. Such extra terms, which act in the isoscalar, color-antitriplet channel, do not remove the degeneracy of the order-parameter sets occurring for equal color chemical potentials and renormalize the coefficients of the fourth order terms so as to reduce the pairing gap. In order to estimate the influence of color neutrality on the gap in the weak coupling limit, we incorporate color chemical potential differences in the gap equation, and find that in the weak coupling limit, such differences do not significantly modify the gap. Although it does not affect the color-flavor-locked phase, the constraint of color neutrality can appreciably raise the free energy of the isoscalar, color-antitriplet phase in the strong coupling regime.

We base our calculations in the weak coupling limit on the field-theoretical description of an ordinary superconductor as advanced by Nambu [22] and Eliashberg [23], who reformulated the original BCS problem in terms of electron-phonon interactions. Such a formalism is useful when the interactions are nonlocal in time, as are the color magnetic interactions in the relativistic plasma. This tool was generalized to a relativistic regime by others in earlier investigations (e.g., Ref. [2]).

We focus in this paper on a system in flavor equilibrium, with no chemical potential differences between flavors (as long as the electrical charge of the quarks is neglected). As a consequence, the energy of the superfluid is invariant under rotations of the equilibrium states in flavor space. The influence of the possible flavor chemical potential differences on the phase diagram, examined by others in earlier investigations [4,24], resolves this degeneracy. Such differences may also give rise to an inhomogeneous phase in which the pairing gap varies periodically in ordinary space. The conditions determining the chemical potential differences between flavors depend, however, sensitively on the extent to which the system is out of flavor equilibrium.

We summarize the main conclusions of this paper. By applying general Ginzburg-Landau theory to the $J = 0$, color and flavor antisymmetric pairing states of superfluid quark matter, we elucidate the phase diagram near T_c in the space of the parameters controlling the fourth order terms in the pairing gap. We find that the phase diagram contains an isoscalar color-antitriplet phase and a color-flavor locked phase, both reached by a second order phase transition, as well as a region of Ginzburg-Landau parameters for which the transition to the superfluid phase is of first order. The detailed structure of the superfluid phase in this regime depends on the nature of the sixth order terms in the expansion. In the limit of weak coupling, the color-flavor locked phase is more favorable than the isoscalar color-antitriplet phase. We also expand the general Ginzburg-Landau approach and the Nambu-Eliashberg formalism to incorporate differences in the chemical potential between colors, required to preserve color-charge neutrality of the system. We find that the color neutrality constraint in the presence of anisotropy of the order parameters in color space tends to suppress the gap.

Color superconductivity in quark matter becomes an astrophysically interesting problem if neutron star interiors are sufficiently dense that they contain quark matter cores [25]. Generally, a quark superfluid in a neutron star would not be electrically neutral since each quark has fractional electric charge; rather it would coexist with electrons (and muons) in such a way as to ensure electric neutrality in the system. Due to the dually charged nature of the quarks, macroscopic manifestations of both color and electromagnetic superconductivity such as Meissner effects, generation of London fields, and vortex formation are expected from magnetic fields and rotations as observed in these celestial objects. We will discuss these issues, which may be relevant to magnetic structure, cooling, and rotational evolution of the neutron stars [4,26], elsewhere [27].

In Sec. II, we construct the generalized Ginzburg-Landau theory including color chemical potential differences, and apply it to color and flavor antisymmetric pairing with $J = 0$. The equilibrium phase diagrams for various values of the parameters controlling the expansion of the free energy are given here. Section III is devoted to deriving the gap equations in the weak coupling limit, into which the color chemical potential differences are incorporated, and to calculations of the energy gap for the two types of optimal pairing states derived in Sec. II. In Sec. IV, we calculate the thermodynamic potential difference between the superfluid and normal phases near T_c from the structure of the energy gap obtained in Sec. III. Our conclusions are given in Sec. V. We use units in which $\hbar = c = k_B = 1$.

II. GENERAL GINZBURG-LANDAU APPROACH

In this section, we apply the general Ginzburg-Landau approach to a quark superfluid that is in an overall color singlet state. We then derive the thermodynamic potentials in terms of the color chemical potential differences. We finally draw the equilibrium phase diagrams near T_c for the parameters characterizing the strong-coupling effects and the color-neutrality constraint.

We consider uniform quark matter of temperature T and baryon chemical potential μ, with number of flavors $N_f = 3$ (u,d,s) and colors $N_c = 3$. We neglect quark masses and electrical charges as well as chiral condensates, and do not take into account at this stage either the color or electromagnetic gauge field, except for virtual gluons mediating interactions between the quarks. We also assume that the system has zero net color charge and satisfies the condition for flavor equilibrium,

$$\mu_{au} = \mu_{ad} = \mu_{as} \equiv \mu_a , \tag{1}$$

where μ_{ai} is the chemical potential for the quarks of color a and flavor i. Thus, the ensemble-averaged color charge densities vanish,

$$\langle \Sigma_i \lambda_{ab}^\alpha \bar{\psi}_{ai}(x) \gamma^0 \psi_{bi}(x) \rangle = 0 , \tag{2}$$

where the ψ_{ai} are the quark fields and λ^{α}_{ab} the color generators. As a consequence,

$$n_R = n_G = n_B \ , \tag{3}$$

with $n_a \equiv \langle \Sigma_i \bar{\psi}_{ai}(x)\gamma^0 \psi_{ai}(x)\rangle$, and

$$n_{ab} \equiv \langle \Sigma_i \bar{\psi}_{ai}(x)\gamma^0 \psi_{bi}(x)\rangle = 0 \tag{4}$$

for $a \neq b$. Condition (1), along with condition (3), relates μ_a to μ as

$$\mu = \sum_a \mu_a \ ; \tag{5}$$

equivalently we write,

$$\tilde{\mu}_{aa} \equiv \mu_a - \frac{\mu}{3} \ . \tag{6}$$

Then

$$\sum_a \tilde{\mu}_{aa} = 0 \ . \tag{7}$$

For $a \neq b$, we introduce the chemical potentials $\tilde{\mu}_{ab}$ associated with n_{ab} in such a way that the grand-canonical Hamiltonian reads $K = H - \sum_{ab}(\delta_{ab}\mu/3 + \tilde{\mu}_{ab})[\sum_i \int d^3x \bar{\psi}_{bi}(x)\gamma^0\psi_{ai}(x)]$. Following Iwasaki and Iwado [28], we can regard these $\tilde{\mu}_{ab}$ as Lagrange multipliers that ensure condition (4). The hermiticity of K requires $\tilde{\mu}_{ab} = \tilde{\mu}^*_{ba}$.

Cooper pairing between quarks, which we assume to be in a channel with zero total angular momentum, is most generally characterized by a $4N_f N_c \times 4N_f N_c$ gap matrix in color, flavor, and Dirac space [19],

$$\Delta(k) = \Delta^{(1)}(k_0, |\mathbf{k}|)\gamma^5 + \Delta^{(2)}(k_0, |\mathbf{k}|)\boldsymbol{\gamma}{\cdot}\hat{\mathbf{k}}\gamma^0\gamma^5 + \Delta^{(3)}(k_0, |\mathbf{k}|)\gamma^0\gamma^5 + \Delta^{(4)}(k_0, |\mathbf{k}|)$$
$$+\Delta^{(5)}(k_0, |\mathbf{k}|)\boldsymbol{\gamma}{\cdot}\hat{\mathbf{k}}\gamma^0 + \Delta^{(6)}(k_0, |\mathbf{k}|)\boldsymbol{\gamma}{\cdot}\hat{\mathbf{k}} + \Delta^{(7)}(k_0, |\mathbf{k}|)\boldsymbol{\gamma}{\cdot}\hat{\mathbf{k}}\gamma^5 + \Delta^{(8)}(k_0, |\mathbf{k}|)\gamma^0 \ , \tag{8}$$

where k is the relative four-momentum between the two quarks forming the Cooper pair, $\hat{\mathbf{k}} \equiv \mathbf{k}/|\mathbf{k}|$, and $\Delta^{(n)}_{abij}$ are the $N_f N_c \times N_f N_c$ matrices denoting the pairing of the quark of color a and flavor i with that of color b and flavor j. The gap Δ is formally related to the pair amplitude $\langle \psi^C(x)\bar{\psi}(y)\rangle$ via [22,23,29]

$$\Delta(k) = ig^2 T \sum_{n \text{ odd}} \int \frac{d^3q}{(2\pi)^3} \int d^4(x - y)e^{iq(x-y)}\gamma^\mu \frac{(\lambda^\alpha)^T}{2}$$
$$\times \left\{ \langle T[\psi^C(x)\bar{\psi}(y)]\rangle \Gamma^{(11)}_{\nu\beta}(q, k) + \langle T[\psi^C(x)\bar{\psi}^C(y)]\rangle \Gamma^{(21)}_{\nu\beta}(q, k) \right\} D^{\alpha\beta}_{\mu\nu}(k - q) \ , \tag{9}$$

where $\psi^C_{ai} \equiv C\bar{\psi}^T_{ai}$ is the charge-conjugate spinor ($C = i\gamma^2\gamma^0$ in the Pauli-Dirac representation), $\Gamma^{(11)}$ is the full quark-quark-gluon vertex, $\Gamma^{(21)}$ is the full antiquark-quark-gluon vertex, D is the full gluon propagator, and the Matsubara frequencies are given by $q_0 = in\pi T$. (In Sec. III we analyze this gap equation in the weak coupling limit; see Eq. (53).) The Pauli principle requires that $\Delta^{(n)}$ satisfy

$$\Delta^{(n)}_{abij}(k_0, |\mathbf{k}|) = \Delta^{(n)}_{baji}(-k_0, |\mathbf{k}|) \ , \qquad n = 1, ..., 6, \tag{10}$$

$$\Delta^{(n)}_{abij}(k_0, |\mathbf{k}|) = -\Delta^{(n)}_{baji}(-k_0, |\mathbf{k}|) \ , \qquad n = 7, 8. \tag{11}$$

For massless quarks the condensates are eigenstates of chirality [30]. Wilson renormalization-group analyses [14] show that the pairing instability occurs between quarks of the same chirality, rather than between the left- and right-handed quarks. We thus ignore the terms $\Delta^{(3)}$, $\Delta^{(6)}$, $\Delta^{(7)}$, and $\Delta^{(8)}$ in Eq. (8), which are associated with quarks of opposite chirality [19]. The $\Delta^{(4)}$ and $\Delta^{(5)}$ terms in Eq. (8) correspond to condensation in the odd-parity channel [19]. Effects of instantons, which prefer even-parity to odd-parity condensates [13], lead us to drop these terms. Since only $\Delta^{(1)}$ and $\Delta^{(2)}$ remain in Eq. (8), it is convenient to introduce

$$\phi_\pm(k_0, |\mathbf{k}|) \equiv \Delta^{(1)}(k_0, |\mathbf{k}|) \mp \Delta^{(2)}(k_0, |\mathbf{k}|) \ , \tag{12}$$

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and rewrite $\Delta(k)$ as

$$\Delta(k) = \gamma^5 [\phi_+(k_0, |\mathbf{k}|)\Lambda^+(\hat{\mathbf{k}}) + \phi_-(k_0, |\mathbf{k}|)\Lambda^-(\hat{\mathbf{k}})] \ . \tag{13}$$

Here, the

$$\Lambda^\pm(\hat{\mathbf{k}}) = \frac{1 \pm \gamma^0 \boldsymbol{\gamma} \cdot \hat{\mathbf{k}}}{2} \tag{14}$$

are energy projection operators for noninteracting massless quarks, and $\phi_\pm$ denotes the quark-quark (antiquark-antiquark) pairing gap.

We divide the difference $\Delta\Omega[\phi_\pm(k_0, |\mathbf{k}|); \tilde{\mu}_{ab}]$ of the thermodynamic potential near T_c of the superfluid phase, Ω_{SF}, from that of the normal phase, Ω_{N}, into two parts:

$$\Delta\Omega = \Omega_0 + \Omega_{\mathrm{CN}} \ , \tag{15}$$

where

$$\Omega_0 = \Delta\Omega[\phi_\pm(k_0, |\mathbf{k}|); \tilde{\mu}_{ab} = 0] \tag{16}$$

has the form of a usual Ginzburg-Landau free energy, and

$$\Omega_{\mathrm{CN}} = \Delta\Omega[\phi_\pm(k_0, |\mathbf{k}|); \tilde{\mu}_{ab}] - \Omega_0 \tag{17}$$

is the correction to Ω_0 induced by the color chemical potential differences in the superfluid (the normal phase satisfies $\tilde{\mu}_{ab} = 0$ because of rotational invariance in color space). In expressing Ω_0 in terms of $\phi_\pm$, we follow the line of argument of Mermin and Stare [10]. Note that the grand-canonical Hamiltonian, described by the chemical potential, $\mu/3$, common to all combinations of spins, flavors, and colors, and zero masses, keeps Ω_0 invariant under global U(1) gauge transformations and rotations in color and flavor space; under global $U(1)$, and special unitary color and flavor rotations of the field operators, $\psi \to e^{i\varphi} U_c U_f \psi$, the $\phi_\pm$ transform as $(\phi_\pm)_{abij} \to e^{-2i\varphi}(\phi_\pm)_{cdlm}(U_c^\dagger)_{ca}(U_c^\dagger)_{db}(U_f^\dagger)_{li}(U_f^\dagger)_{mj}$. Taking into account the condition (10) imposed by the Pauli principle, we obtain, up to $\mathcal{O}(\Delta^4)$,

$$\begin{aligned} \Omega_0 = \ & \alpha^+ \mathrm{Tr}(\phi_+^\dagger \phi_+)_F + \beta_1^+ [\mathrm{Tr}(\phi_+^\dagger \phi_+)_F]^2 + \beta_2^+ \mathrm{Tr}[(\phi_+^\dagger \phi_+)^2]_F \\ & + \alpha^- \mathrm{Tr}(\phi_-^\dagger \phi_-)_F + \beta_1^- [\mathrm{Tr}(\phi_-^\dagger \phi_-)_F]^2 + \beta_2^- \mathrm{Tr}[(\phi_-^\dagger \phi_-)^2]_F \ . \end{aligned} \tag{18}$$

Due to the relation, $\Lambda^\pm \Lambda^\mp = 0$, the contributions of the quark-quark condensates separate from those of the antiquark-antiquark condensates. The subscript "F" denotes the pairing gap evaluated for the quark or antiquark quasiparticle momenta $|\mathbf{k}|$ equal to the quark Fermi momentum, k_F.

In general, the energies of antiquark excitations, $\sim \left((|\mathbf{k}| + k_F)^2 + \phi_-^\dagger \phi_-\right)^{1/2}$, are much larger than the energies of quark excitations, $\sim \left((|\mathbf{k}| - k_F)^2 + \phi_+^\dagger \phi_+\right)^{1/2}$. Thus in the temperature region near T_c where $\mu/3$ is large compared with the magnitude of the energy gap, the amplitudes for virtual excitations of antiquark quasiparticles with $|\mathbf{k}|$ near k_F are smaller than those for excitations of quark quasiparticles by a factor $\sim \mathcal{O}(|\phi_+(T \sim T_c)|/\mu)$; as a consequence, α^+ and β_i^+ dominate over α^- and β_i^-, and we set $\alpha^- = \beta_i^- = 0$.

The next step is to express the correction term Ω_{CN} in terms of ϕ_+ and $\tilde{\mu}_{ab}$. We need retain contributions up to second order in $\tilde{\mu}_{ab}$. Since $n_a = -\partial\Omega_{\mathrm{SF}}/\partial\mu_a$ and $n_{ab} = -\partial\Omega_{\mathrm{SF}}/\partial\tilde{\mu}_{ba}$ for $a \neq b$, the conditions for color neutrality, Eqs. (3) and (4), can be written as

$$\frac{\partial\Delta\Omega}{\partial\tilde{\mu}_{RR}} = \frac{\partial\Delta\Omega}{\partial\tilde{\mu}_{GG}} = \frac{\partial\Delta\Omega}{\partial\tilde{\mu}_{BB}} \ , \tag{19}$$

and

$$\frac{\partial\Delta\Omega}{\partial\tilde{\mu}_{ab}} = 0 \tag{20}$$

for $a \neq b$.

Up to $\mathcal{O}(\tilde{\mu}_{ab}\phi_+^2)$ and $\mathcal{O}(\tilde{\mu}_{ab}^2)$, the only terms in Ω_{CN} that are invariant under global U(1) gauge transformations and special unitary transformations in flavor space are proportional to $\mathrm{Tr}\tilde{\mathcal{M}}^2$, $\mathrm{Tr}(\phi_+^\dagger \phi_+ \tilde{\mathcal{M}})_F$, and $\mathrm{Tr}(\phi_+ \phi_+^\dagger \tilde{\mathcal{M}})_F$, where

$$\tilde{\mathcal{M}}_{abij} = \delta_{ij}\tilde{\mu}_{ab} ; \tag{21}$$

in deducing this structure we have used the relations $\mathrm{Tr}(\phi_+\phi_+^\dagger\tilde{\mathcal{M}}^T)_F = \mathrm{Tr}(\phi_+^\dagger\phi_+\tilde{\mathcal{M}})_F$ and $\mathrm{Tr}(\phi_+^\dagger\phi_+\tilde{\mathcal{M}}^T)_F = \mathrm{Tr}(\phi_+\phi_+^\dagger\tilde{\mathcal{M}})_F$, derived from $(\phi_+^T)_F = (\phi_+)_F$ (see Eq. (10)). The term proportional to $\mathrm{Tr}(\phi_+\phi_+^\dagger\tilde{\mathcal{M}})_F$, which does not occur in the weak coupling limit (see Eq. (121)), always has a zero coefficient, even in the strong coupling regime. The reason is that this term, of second order in the gap, is not affected by the dependence of the pairing interaction on the pairing gap, which can only induce terms of at least fourth order in the gap. We thus write

$$\Omega_{\mathrm{CN}} = \sigma\mathrm{Tr}\tilde{\mathcal{M}}^2 + \chi\mathrm{Tr}(\phi_+^\dagger\phi_+\tilde{\mathcal{M}})_F . \tag{22}$$

The first term on the right side of Eq. (22) comes from the change in the normal-fluid free-energy for non-zero $\tilde{\mathcal{M}}$, while the second is the correction to the $\mathcal{O}(\phi_+^2)$ term in Ω_0. Because of color neutrality, σ and χ appear, as we shall see, only in the dimensionless combination χ^2/σ in the energy gap and the condensation energy.

In the remainder of this section, we consider condensates that are antisymmetric in color and flavor. In these condensates, Cooper pairing is characterized by the products between the color-antitriplet states $[(|RG\rangle - |GR\rangle)/\sqrt{2}, (|GB\rangle - |BG\rangle)/\sqrt{2}, (|BR\rangle - |RB\rangle)/\sqrt{2}]$ and the flavor-antitriplet states $[(|ud\rangle - |du\rangle)/\sqrt{2}, (|ds\rangle - |sd\rangle)/\sqrt{2}, (|su\rangle - |us\rangle)/\sqrt{2}]$.

For total angular momentum $J = 0$, the Pauli principle constraint (10) requires that antisymmetry of pairing in color space be accompanied by antisymmetry of pairing in flavor space. Such pairing states can occur in the weak coupling limit, because one-gluon exchange in the color antitriplet channel is attractive [2]. The gap is characterized by the ansatz,

$$(\phi_+)_{abij} = \epsilon_{ijl}\epsilon_{abc}A_{lc} , \tag{23}$$

where $A \equiv (\mathbf{d}_R, \mathbf{d}_G, \mathbf{d}_B)$ is a general complex matrix. $A_{ia}, = (\mathbf{d}_a)_i$, denotes the gap for pairing between quarks of colors b and c with $a \neq b \neq c$ and of flavors j and l with $i \neq j \neq l$.

Substituting Eq. (23) into Eqs. (18) and (22), we derive the Ginzburg-Landau form of Ω_0 and Ω_{CN} in terms of A or $\mathbf{d}_a$:

$$\begin{aligned}\Omega_0 &= \bar{\alpha}\mathrm{Tr}(A^\dagger A)_F + \beta_1[\mathrm{Tr}(A^\dagger A)_F]^2 + \beta_2\mathrm{Tr}[(A^\dagger A)^2]_F \\ &= \bar{\alpha}\lambda + (\beta_1 + \beta_2\Upsilon)\lambda^2 ,\end{aligned} \tag{24}$$

with

$$\bar{\alpha} \equiv 4\alpha^+ , \quad \beta_1 \equiv 16\beta_1^+ + 2\beta_2^+ , \quad \beta_2 \equiv 2\beta_2^+ , \tag{25}$$

$$\lambda \equiv \sum_a |\mathbf{d}_a|_F^2 , \quad \Upsilon \equiv \frac{1}{\lambda^2}\sum_{ab} |\mathbf{d}_a^* \cdot \mathbf{d}_b|_F^2 , \tag{26}$$

and

$$\Omega_{\mathrm{CN}} = 3\sigma\sum_{ab} |\tilde{\mu}_{ab}|^2 - 2\chi\sum_{ab}(\mathbf{d}_a^* \cdot \mathbf{d}_b)_F\tilde{\mu}_{ab} . \tag{27}$$

One can readily show that under a global $U(1)$, color, and flavor rotation of the field operators, $\psi \to e^{i\varphi}U_cU_f\psi$, A transforms as $A_{ia} \to e^{-2i\varphi}(U_c)_{ab}(U_f)_{ij}A_{jb}$. The second order and fourth order terms in A, included in Ω_0, Eq. (24), are the only invariants under these transformations. In Ω_{CN}, Eq. (27), the term of linear order in $\tilde{\mu}_{ab}$ is not affected by global U(1) gauge transformations, $\mathbf{d}_a \to e^{-2i\varphi}\mathbf{d}_a$, and flavor rotations, $\mathbf{d}_a \to U_f\mathbf{d}_a$.

Note that Υ, which is dimensionless, ranges from 1/3 to 1. For $\Upsilon = 1/3$, the configurations for $\mathbf{d}_a$ are determined by

$$\mathbf{d}_R^* \cdot \mathbf{d}_G = \mathbf{d}_G^* \cdot \mathbf{d}_B = \mathbf{d}_B^* \cdot \mathbf{d}_R = 0 , \quad |\mathbf{d}_R|^2 = |\mathbf{d}_G|^2 = |\mathbf{d}_B|^2 ; \tag{28}$$

for $\Upsilon = 1$, the vectors $\mathbf{d}_R$, $\mathbf{d}_G$, and $\mathbf{d}_B$ are all parallel:

$$\mathbf{d}_R \parallel \mathbf{d}_G \parallel \mathbf{d}_B . \tag{29}$$

In the weak coupling limit, which we calculate in Sec. IV, the coefficients $\bar{\alpha}$, β_1, β_2, σ, and χ reduce to

$$\bar{\alpha} = 4N(\mu/3) \ln\left(\frac{T}{T_c}\right) \ , \tag{30}$$

$$\beta_1 = \frac{7\zeta(3)}{8(\pi T_c)^2} N(\mu/3) \ , \tag{31}$$

$$\beta_2 = \frac{7\zeta(3)}{8(\pi T_c)^2} N(\mu/3) \ , \tag{32}$$

$$\sigma = -N(\mu/3) \ , \tag{33}$$

$$\chi = \frac{3}{\mu} \ln\left(\frac{3T_c}{\mu}\right) N(\mu/3) \ , \tag{34}$$

where

$$N(\mu/3) = \frac{1}{2\pi^2} \left(\frac{\mu}{3}\right)^2 \tag{35}$$

is the ideal gas density of states at the Fermi surface, and the zeta function $\zeta(3) = 1.2020\ldots$.

Two effects beyond weak coupling must in general be taken into account. The first is the modification of the pairing interaction due to the pairing gap, which modifies the coefficients β_1 and β_2 of the fourth order terms in A. The second is radiative corrections by the normal medium [21] – i.e., quark self-energy, gluon polarization, and quark-quark-gluon and three-gluon vertex corrections – which modify the coefficients Eqs. (30)-(34), mainly through their dependence on $N(\mu/3)$ and T_c.

We proceed to minimize the Ginzburg-Landau free energy, Eqs. (24) and (27), with respect to the $(\mathbf{d}_a)_F$. To elucidate effects of the color neutrality, it is instructive to start with the optimal expressions for the energy gap and the condensation energy in the case in which the $\tilde{\mu}_{ab}$ and hence $\tilde{\mathcal{M}}$ and $\Omega_{\rm CN}$ vanish. From Ω_0, written up to second order in λ, we find that thermodynamic stability requires $\bar{\beta} \equiv \beta_1 + \beta_2 \Upsilon > 0$ for $1/3 \leq \Upsilon \leq 1$. It is straightforward to show that in this stable region, only two phases occur: $\Upsilon = 1/3$ for $\beta_2 > 0$ and $\Upsilon = 1$ for $\beta_2 < 0$. We discuss the physics of the region where $\bar{\beta} < 0$ below.

The order parameters in the $\Upsilon = 1$ phase satisfy condition (29). All such order parameters lead to states degenerate in energy. Note that in this order-parameter set, any state is identical to an isoscalar, color-antiplet state characterized by $(\mathbf{d}_a)_i \propto \delta_{is}$ (s, the strange flavor), or equivalently,

$$(\phi_+)_{abij} = \epsilon_{abc}\epsilon_{ijs}(\mathbf{d}_c)_s \ , \tag{36}$$

to within a constant phase factor and a special unitary transformation in flavor space. In the corresponding condensate, the u and d quarks are paired in an isosinglet state. We shall refer to the $\Upsilon = 1$ phase as the isoscalar phase, even though it contains order parameters having the other orientations in complex flavor space than the s direction. The magnitude of the gap in the isoscalar phase is,

$$\lambda = -\frac{\bar{\alpha}}{2(\beta_1 + \beta_2)} \ , \tag{37}$$

with condensation energy,

$$\Delta\Omega = -\frac{\bar{\alpha}^2}{4(\beta_1 + \beta_2)} \ . \tag{38}$$

The order parameters in the $\Upsilon = 1/3$ phase fulfill condition (28), and lead to degenerate states. This phase is the color-flavor locked phase [15]; the condensate in this phase is characterized by its symmetry under simultaneous exchange of color and flavor. States belonging to this phase transform into one another under global U(1) gauge transformations and special unitary transformations in flavor space. The simplest among these states is described by $(\mathbf{d}_a)_i \propto \delta_{ai}$ and $(\mathbf{d}_R)_u = (\mathbf{d}_G)_d = (\mathbf{d}_B)_s \equiv \kappa_A$; the corresponding gap matrix is given explicitly by

$$(\phi_+)_{abij} = \kappa_A(\delta_{ai}\delta_{bj} - \delta_{aj}\delta_{bi}) \ . \tag{39}$$

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For the color-flavor locked phase, we obtain the magnitude of the gap,

$$\lambda = -\frac{\bar{\alpha}}{2(\beta_1 + \beta_2/3)} \ , \tag{40}$$

and the condensation energy,

$$\Delta\Omega = -\frac{\bar{\alpha}^2}{4(\beta_1 + \beta_2/3)} \ . \tag{41}$$

In the original definition [15] of the color-flavor locked state, in addition to the gap, κ_A, in the color and flavor antisymmetric channel, a gap matrix $\kappa_S(\delta_{ai}\delta_{bj} + \delta_{aj}\delta_{bi})$ arises in the color and flavor symmetric channel. In the weak coupling limit, the gap κ_A can be generated by the attractive one-gluon exchange interaction in the color-antitriplet channel. Nonzero values of κ_S, on the other hand, are not driven by the one-gluon exchange interaction since it is repulsive in the color symmetric channel; they are $\mathcal{O}(g)$ correction to κ_A that ensures the existence of the $\kappa_A \neq 0$ solution to the relevant weak coupling gap equation (see Ref. [11]). This implies that the gap κ_S can be ignored in the temperature region near the onset of the pairing.

We now construct, in the β_1-β_2 plane, the phase diagram exhibiting the more stable pairing state of the isoscalar, and color-flavor locked phases. We first address the question of whether or not T_c, the temperature at which the pairing instability of the normal phase occurs, is the same for these phases. The onset of this instability is controlled solely by the pairing interactions in the normal phase between quark quasiparticles with zero total momentum. The corresponding amplitudes depend on the color, flavor, and quantum numbers such as the total angular momentum, chirality, and parity of the quarks involved in the pairing [2], and for the two phases considered here, the quark-quark pairs have antisymmetric structures in color and flavor, the same chirality, even parity, and $J = 0$ at the onset of the pairing instability. We thus conclude that no difference in T_c arises between these phases. The amplitudes in the normal system driving the instability do not distinguish between the final possible paired states.

Figure 1 shows the map of the isoscalar and color-flavor locked phases in the β_1-β_2 plane, for the case in which $\tilde{\mathcal{M}}$, or equivalently, Γ, is set equal to zero. The regions subtended by these phases are restricted by the region in which $\bar{\beta} > 0$, for $1/3 \leq \Upsilon \leq 1$, is violated. By comparing the condensation energies Eqs. (38) and (41), we find that in the weak coupling limit, described here by Eqs. (30)-(32), the color-flavor locked state is favored over the isoscalar channel, a result consistent with the conclusion drawn from the weak coupling analyses [11,12] at zero temperature.
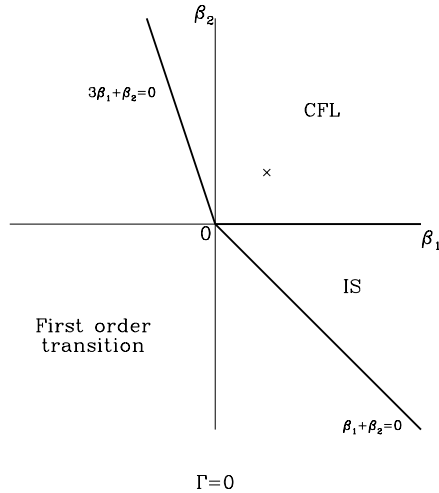

Fig. 1. Phase diagram in the Ginzburg-Landau regime, showing regions where the isoscalar (IS) and color-flavor locked (CFL) phases are favored, when the constraint of color neutrality does not affect the free energy, $\Gamma = 0$. The β_1, β_2 are the fourth order coefficients in the Ginzburg-Landau free energy, Eq. (24), and Γ (Eqs. (46) and (27)) describes effects of the constraint of color neutrality. The cross indicates the weak coupling limit. In the region of first order transitions, the overall fourth order coefficient, $\bar{\beta}$, is not positive definite.

The region in which $\bar{\beta} < 0$ corresponds to a *first order* phase transition from the normal to the superfluid state at a temperature greater than T_c^0 at which $\bar{\alpha}(T_c^0)$ vanishes, and the normal state becomes unstable against Cooper pairing. The situation is similar to condensed matter systems in which the Ginzburg-Landau free energy contains a term cubic in the magnitude of the order parameter, and also similar to the situation in which the chiral phase transition in QCD with two-flavor massless quarks changes from second order to first order at a tricritical point in the μ versus T plane [31]. To see this structure, we add to the Ginzburg-Landau expansion up to fourth order in A, Eq. (24), the sum of the sixth order invariants,

$$\Omega_6 = (\bar{\gamma}_1 + \bar{\gamma}_2 \Upsilon + \bar{\gamma}_3 \xi)\lambda^3 , \tag{42}$$

where λ and Υ are given by Eq. (26), and

$$\xi \equiv \frac{1}{\lambda^3} \sum_{abc} (\mathbf{d}_a^* \cdot \mathbf{d}_b)_F (\mathbf{d}_b^* \cdot \mathbf{d}_c)_F (\mathbf{d}_c^* \cdot \mathbf{d}_a)_F . \tag{43}$$

Let us assume that the overall coefficient of λ^3, $\bar{\gamma}_1 + \bar{\gamma}_2 \Upsilon + \bar{\gamma}_3 \xi \equiv \bar{\gamma}$, is positive. Then, the local minimum of $\Omega_0 + \Omega_6$ is reached at a nonzero value of λ that can be calculated as $\lambda_0 = |\bar{\beta}|/3\bar{\gamma} + [(\bar{\beta}/3\bar{\gamma})^2 - \bar{\alpha}/3\bar{\gamma}]^{1/2}$. The critical temperature T_c can be determined from the condition $\Omega_0(\lambda_0) + \Omega_6(\lambda_0) = 0$, or equivalently, $\bar{\alpha}(T_c) = \bar{\beta}^2/4\bar{\gamma}$, where we can ignore the temperature dependence of $\bar{\beta}$ and $\bar{\gamma}$. Note that T_c is greater than T_c^0. We thus find that $\lambda_0 = |\bar{\beta}|/2\bar{\gamma}$ at $T = T_c$, i.e., the pairing gap is discontinuous at the transition point. We remark that this argument is only applicable to the case in which $T_c - T_c^0 \ll T_c^0$. In the case that $\bar{\beta} < 0$ and $\bar{\gamma} < 0$, one must go to higher order to determine the critical temperature and the discontinuity of the order parameter at the transition.

We turn to the Ginzburg-Landau energy, Eqs. (24) and (27), for a color-singlet system. As we shall see, the constraint of color neutrality acts to modify the pairing gap without removing the degeneracy of the order parameters occurring when the differences in color chemical potentials vanish. Generally, the deviations of the chemical potential differences $\tilde{\mu}_{ab}$ from zero can be determined from the color neutrality conditions (19) and (20) as

$$\tilde{\mu}_{ab} = \frac{\chi}{9\sigma} [3(\mathbf{d}_a \cdot \mathbf{d}_b^*)_F - \delta_{ab} \sum_c |\mathbf{d}_c|_F^2] . \tag{44}$$

Substitution of this expression into Eqs. (24) and (27) leads to the Ginzburg-Landau form,

$$\Delta\Omega = \bar{\alpha}\lambda + [(\beta_1 - \Gamma) + (\beta_2 + 3\Gamma)\Upsilon]\lambda^2 , \tag{45}$$

with

$$\Gamma \equiv -\frac{1}{\sigma} \left(\frac{\chi}{3}\right)^2 . \tag{46}$$

The coefficient Γ is positive definite as long as the color-singlet system is thermodynamically stable against color fluctuations, i.e., $(\partial^2 \Delta\Omega / \partial\tilde{\mu}_{ab}\partial\tilde{\mu}_{ba})_{(\mathbf{d}_c)_F} = 6\sigma < 0$. This condition holds in the weak coupling limit, Eq. (33); vacuum polarization effects in the low-density regime, which lead to color antiscreening, or positive color susceptibility, $-\partial^2\Omega/\partial\tilde{\mu}_{ab}\partial\tilde{\mu}_{ba} > 0$, are in the direction to preserve this condition. We thus assume $\Gamma > 0$ for arbitrary coupling constant g.

Note that Eq. (45) is identical with Eq. (24) except that β_1 and β_2 are replaced by $\beta_1 - \Gamma$ and $\beta_2 + 3\Gamma$, i.e., inclusion of the color neutrality results in a renormalization of β_1 and β_2. This is due to the fact that the constraint of color neutrality itself is invariant under color rotations and thus chemical potential differences between colors yield the fourth order terms invariant with respect to rotations in color space.

We then obtain from Eq. (45) the two sets of optimal order parameters, $\Upsilon = 1/3$ (the color-flavor locked phase) for $\beta_2 + 3\Gamma > 0$ and $\Upsilon = 1$ (the isoscalar phase) for $\beta_2 + 3\Gamma < 0$. The resultant magnitude of the gap in the isoscalar phase can be written as

$$\lambda = -\frac{\bar{\alpha}}{2(\beta_1 + \beta_2 + 2\Gamma)} , \tag{47}$$

with condensation energy,

$$\Delta\Omega = -\frac{\bar{\alpha}^2}{4(\beta_1 + \beta_2 + 2\Gamma)} . \tag{48}$$

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We observe from the energy gap and condensation energy, Eqs. (47) and (48), that λ and $|\Delta\Omega|$ are suppressed at fixed β_1 and β_2 by the parameter Γ characterizing the color neutrality. This suppression, together with $\tilde{\mathcal{M}} \neq 0$ as can be found from substitution of condition (29) into Eq. (44), comes from the fact that the condensate is intrinsically anisotropic in color space, as we shall see in Sec. III A. We remark in passing that in the weak coupling limit, where only contributions of leading order in $3T_c/\mu$ remain, Γ is dominated by β_1 and β_2 and hence does not significantly affect the gap. In the color-flavor locked phase, on the other hand, the gap size and the condensation energy of the color-flavor locked phase are still given by Eqs. (40) and (41). For this phase, we can find $\tilde{\mathcal{M}} = 0$ from substitution of condition (28) into Eq. (44). This vanishing reflects the fact that the condensate in the color-flavor locking is isotropic in color space, as suggested by condition (28).

We conclude this section by examining the influence of the requirement of color neutrality on the phase diagram in the β_1-β_2 plane. As we see in Fig. 2, the phase diagram for $\Gamma > 0$, this requirement shifts the isoscalar phase by Γ in the β_1 direction and -3Γ in the β_2 direction. With increasing Γ, this phase, in which the order parameters are anisotropic in color space, moves farther away from the weak coupling point, (β_1, β_2) given by Eqs. (31) and (32). With increasing Γ, the color-flavor locked phase begins to cover the region occupied by the isoscalar phase.

To answer the question of which phase is more stable at low densities will require understanding in detail how strong coupling effects, which develop as the baryon chemical potential μ is lowered, modify β_1, β_2, and Γ with increasing g. These effects are too uncertain for us to be able to predict in general the more stable state near T_c.
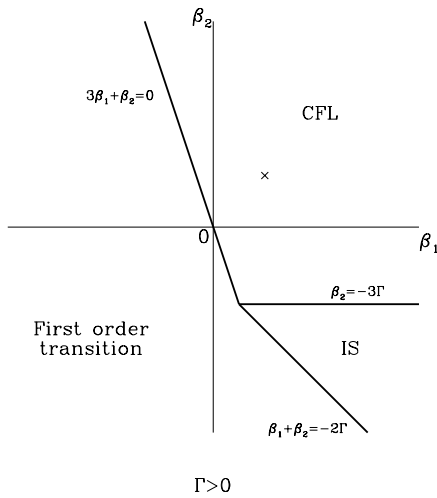

Fig. 2. Same as Fig. 1 with the constraint of color neutrality, described by $\Gamma > 0$.

III. GAP EQUATION

We turn now to deriving the high-density asymptotic form of the thermodynamic potential difference $\Delta\Omega$ near T_c, as we used in the previous section. The gap equation in the weak coupling limit provides information on the overall temperature dependence of the gap and the quasiparticle structures, in contrast to the Ginzburg-Landau approach developed in the previous section, which concentrates on limited temperature region near T_c and coarse-grained features of the superfluid. For the moment, we consider an arbitrary color neutral condensate of the form (13). For the two optimal pairing states considered in Sec. II, we obtain the behavior of the energy gap from the finite-temperature gap equation, with the constraint of color neutrality. In Sec. IV, we derive the coefficients given by Eqs. (30)-(34) from the structure of the energy gap.

In obtaining the gap equation in the weak coupling limit, it is convenient to first introduce the notation for the quark field, (ψ_{ai}, ψ_{ai}^C). We then write a self-consistent Schwinger-Dyson equation to determine the proper self energy $\Sigma(k)$ up to $\mathcal{O}(g^2)$ [2]:

$$\Sigma(k) = -g^2 T \int \frac{d^3q}{(2\pi)^3} \sum_{n \text{ odd}} D_{\mu\nu}^{\alpha\beta}(k-q)$$

$$\times \begin{pmatrix} \gamma^\mu \lambda^\alpha/2 & 0 \\ 0 & -\gamma^\mu(\lambda^\alpha)^T/2 \end{pmatrix} G(q) \begin{pmatrix} \gamma^\nu \lambda^\beta/2 & 0 \\ 0 & -\gamma^\nu(\lambda^\beta)^T/2 \end{pmatrix} , \tag{49}$$

where the summation is over Matsubara frequencies, $q_0 = in\pi T$, with n odd for fermions,

$$G(k) \equiv \begin{pmatrix} G^{(11)}(k) & G^{(12)}(k) \\ G^{(21)}(k) & G^{(22)}(k) \end{pmatrix} = \begin{pmatrix} \gamma k + \gamma^0 \mathcal{M} & \tilde{\Delta}(k) \\ \Delta(k) & \gamma k - \gamma^0 \mathcal{M}^T \end{pmatrix}^{-1} \tag{50}$$

is the quark propagator with normal state Hartree-Fock contributions ignored, $\tilde{\Delta} = \gamma^0 \Delta^\dagger \gamma^0$ and $\mathcal{M}_{abij} = \delta_{ij}(\delta_{ab}\mu/3 + \tilde{\mu}_{ab})$, and $D_{\mu\nu}^{\alpha\beta}$ is the gluon propagator, specified below.

The self-energy and quark propagator are related by

$$G^{-1}(k) = G_0^{-1}(k) - \Sigma(k) , \tag{51}$$

where

$$G_0(k) = \begin{pmatrix} \gamma k + \gamma^0 \mathcal{M} & 0 \\ 0 & \gamma k - \gamma^0 \mathcal{M}^T \end{pmatrix}^{-1} \tag{52}$$

is the noninteracting quark propagator.

The off-diagonal component of Eq. (49) yields the gap equation in the weak coupling limit,

$$\Delta(k) = -g^2 T \int \frac{d^3q}{(2\pi)^3} \sum_{n \text{ odd}} D_{\mu\nu}^{\alpha\beta}(k-q)\gamma^\mu \frac{(\lambda^\alpha)^T}{2} G^{(21)}(q)\gamma^\nu \frac{\lambda^\beta}{2} , \tag{53}$$

where

$$G^{(21)}(q) = (\gamma q - \gamma^0 \mathcal{M}^T)^{-1}\Delta(q) \left[\tilde{\Delta}(q)(\gamma q - \gamma^0 \mathcal{M}^T)^{-1}\Delta(q) - (\gamma q + \gamma^0 \mathcal{M}) \right]^{-1} \tag{54}$$

is the 21-component of $G(q)$. The summation over n on the right side of Eq. (53) leads to the self-consistent gap equation,

$$\Delta(\omega_\pm(\mathbf{k}), \mathbf{k}) = -\frac{1}{4}g^2 \sum_{\eta=\pm} \eta \int \frac{d^3q}{(2\pi)^3} D_{\mu\nu}^{\alpha\beta}(\varepsilon_\pm(\mathbf{k}) - \eta\varepsilon_\eta(\mathbf{q}), \mathbf{k} - \mathbf{q})$$

$$\times \gamma^\mu \frac{(\lambda^\alpha)^T}{2} \left[\hat{\Delta}(\omega_\eta(\mathbf{q}), \mathbf{q})\varepsilon_\eta^{-1}(\mathbf{q}) \tanh\left(\frac{\omega_\eta(\mathbf{q})}{2T}\right) \right] \gamma^\nu \frac{\lambda^\beta}{2} , \tag{55}$$

where the (on-shell) gap matrix $\hat{\Delta}(k)$ is given by,

$$\hat{\Delta}(k) = \Lambda^-(\hat{\mathbf{k}})\gamma^0 \Delta(k)\Lambda^+(\hat{\mathbf{k}})\gamma^0 , \tag{56}$$

with the Hermitean frequency matrix

$$\omega_\pm(\mathbf{k}) = \pm\varepsilon_\pm(\mathbf{k}) - i\text{Im}\mathcal{M} , \tag{57}$$

and

$$(\varepsilon_\eta(\mathbf{k}))_{abij} = \left(\delta_{ij} \sum_c (\delta_{ac}(|\mathbf{k}| - \mu/3) - \text{Re}\tilde{\mu}_{ac})(\delta_{cb}(|\mathbf{k}| - \mu/3) - \text{Re}\tilde{\mu}_{cb}) \right.$$

$$\left. + \frac{1}{2} \sum_{cl\mu\nu} \tilde{\Delta}_{acil}^{\mu\nu}(\omega_\eta(\mathbf{k}), \mathbf{k})\hat{\Delta}_{cblj}^{\nu\mu}(\omega_\eta(\mathbf{k}), \mathbf{k}) \right)^{1/2} \tag{58}$$

the energy matrix for quark quasiparticles. Here, we have used $\mathcal{M}\mathcal{M}^T = (\text{Re}\mathcal{M})^2 + \mathcal{O}(\tilde{\mathcal{M}}^2)$ and made the approximation in $G^{(21)}(q)$, Eq. (54), that $(\gamma q - \gamma^0 \mathcal{M}^T)^{-1}\Delta(q)(\gamma q - \gamma^0 \mathcal{M}^T) \approx \tilde{\Delta}^\dagger(q)$. This approximation is sufficient to

describe the gap matrix up to first order in $\tilde{\mathcal{M}}$, since $||\mathbf{q}| - \mu/3||\tilde{\mu}_{ab}|$ is much smaller than the square of the gap magnitude in the momentum region where the gap is appreciable (as we shall see in Sec. III A). We have also disregarded the contributions of antiquark quasiparticles to the integral over $\mathbf{q}$ in Eq. (55); in the weak coupling regime, these contributions are suppressed by one power of Δ/μ compared with the quark quasiparticle contributions since the integral is dominated by the region immediately close to $|\mathbf{q}| = \mu/3$. Using the definition (12), we obtain

$$\hat{\Delta}(k) = -\gamma^5 \phi_+(k) \Lambda^-(\hat{\mathbf{k}}) , \tag{59}$$

and the quasiparticle energies

$$\varepsilon_\eta(\mathbf{k}) = [(|\mathbf{k}| - \mathrm{Re}\mathcal{M})^2 + \phi_+^\dagger(\omega_\eta(\mathbf{k}), \mathbf{k})\phi_+(\omega_\eta(\mathbf{k}), \mathbf{k})]^{1/2} . \tag{60}$$

Equation (55) can now be rewritten in terms of ϕ_+ as

$$\phi_+(\omega_\pm(\mathbf{k}), \mathbf{k}) = -\frac{g^2}{8} \sum_{\eta=\pm} \eta \int \frac{d^3q}{(2\pi)^3} D_{\mu\nu}^{\alpha\beta}(\varepsilon_\pm(\mathbf{k}) - \eta\varepsilon_\eta(\mathbf{q}), \mathbf{k} - \mathbf{q}) \mathrm{Tr}[\gamma^\mu \Lambda^-(\hat{\mathbf{q}})\gamma^\nu \Lambda^+(\hat{\mathbf{k}})]$$

$$\times \frac{1}{4}(\lambda^\alpha)^T \phi_+(\omega_\eta(\mathbf{q}), \mathbf{q})\varepsilon_\eta^{-1}(\mathbf{q}) \tanh\left(\frac{\omega_\eta(\mathbf{q})}{2T}\right) \lambda^\beta . \tag{61}$$

The gap equation (61) contains the color chemical potential differences $\tilde{\mu}_{ab}$, which are determined by the color neutrality conditions, Eqs. (3) and (4), written in terms of n_{ab} with $n_a \equiv n_{aa}$; these n_{ab}, characterizing the color charge densities, are given by

$$n_{ab} = T \sum_i \int \frac{d^3q}{(2\pi)^3} \sum_{n \ \mathrm{odd}} \mathrm{Tr}[\gamma^0 G_{abii}^{(11)}(q)] , \tag{62}$$

where

$$G^{(11)}(q) = \left[(\gamma q + \gamma^0 \mathcal{M}) - \tilde{\Delta}(q)(\gamma q - \gamma^0 \mathcal{M}^T)^{-1}\Delta(q)\right]^{-1} \tag{63}$$

is the 11-component of the propagator $G(q)$. In terms of $\phi_\pm$,

$$G^{(11)}(q) = (\gamma q - \gamma^0 \mathcal{M}^T) \left\{[(q_0 + i\mathrm{Im}\mathcal{M})^2 - (|\mathbf{q}| + \mathrm{Re}\mathcal{M})^2 - \phi_-^\dagger(q)\phi_-(q)]^{-1}\Lambda^+(\hat{\mathbf{q}})\right.$$

$$\left. +[(q_0 + i\mathrm{Im}\mathcal{M})^2 - (|\mathbf{q}| - \mathrm{Re}\mathcal{M})^2 - \phi_+^\dagger(q)\phi_+(q)]^{-1}\Lambda^-(\hat{\mathbf{q}})\right\} , \tag{64}$$

where we have again used the approximation $(\gamma q - \gamma^0 \mathcal{M}^T)^{-1}\Delta(q)(\gamma q - \gamma^0 \mathcal{M}^T) \approx \tilde{\Delta}^\dagger(q)$. As in the derivation of the gap equation (55), this approximation is sufficient to determine n_{ab} up to first order in $\tilde{\mathcal{M}}$. Substituting Eq. (64) into Eq. (62), we find

$$n_{ab} = T \sum_i \int \frac{d^3q}{(2\pi)^3} \sum_{n \ \mathrm{odd}} 2 \left\{(q_0 - \mathcal{M}^T - |\mathbf{q}|)[(q_0 + i\mathrm{Im}\mathcal{M})^2 - (|\mathbf{q}| + \mathrm{Re}\mathcal{M})^2 - \phi_-^\dagger(q)\phi_-(q)]^{-1}\right.$$

$$\left. +(q_0 - \mathcal{M}^T + |\mathbf{q}|)[(q_0 + i\mathrm{Im}\mathcal{M})^2 - (|\mathbf{q}| - \mathrm{Re}\mathcal{M})^2 - \phi_+^\dagger(q)\phi_+(q)]^{-1}\right\}_{abii} . \tag{65}$$

The Matsubara frequency summation in this equation yields the usual BCS expression,

$$n_{ab} = \sum_i \int \frac{d^3q}{(2\pi)^3} \{1 - u_+^2(\mathbf{q})[1 - 2f(\omega_+(\mathbf{q}))] + v_-^2(\mathbf{q})[1 - 2f(-\omega_-(\mathbf{q}))]$$

$$+\mathcal{O}(\phi_-^\dagger \phi_-/\mu^2)\}_{abii} , \tag{66}$$

where

$$f(\varepsilon) = \frac{1}{e^{\varepsilon/T} + 1} \tag{67}$$

is the distribution function for quark quasiparticles, and the BCS coherence factors are given by

$$u_\eta^2(\mathbf{q}) = \frac{1}{2}[\varepsilon_\eta(\mathbf{q}) - \mathrm{Re}\mathcal{M} + |\mathbf{q}|]\varepsilon_\eta^{-1}(\mathbf{q}) , \quad v_\eta^2(\mathbf{q}) = \frac{1}{2}[\varepsilon_\eta(\mathbf{q}) + \mathrm{Re}\mathcal{M} - |\mathbf{q}|]\varepsilon_\eta^{-1}(\mathbf{q}) , \tag{68}$$

with $u_\eta^2(\mathbf{q}) + v_\eta^2(\mathbf{q}) = 1$. In obtaining $\tilde{\mathcal{M}}$ up to leading order in ϕ_+, Eq. (66) can be simplified as

$$n_{ab} = 2\sum_i \int \frac{d^3q}{(2\pi)^3}\{f(|\mathbf{q}| - \mathcal{M}) - f(|\mathbf{q}| - \mu/3)$$
$$+[u_+^2(\mathbf{q})f(\varepsilon_+(\mathbf{q})) + v_+^2(\mathbf{q})(1 - f(\varepsilon_+(\mathbf{q})))]_{\tilde{\mathcal{M}}=0}\}_{abii} , \tag{69}$$

where we have used the fact that $\varepsilon_+(\mathbf{q})|_{\mathrm{Im}\mathcal{M}=0} = \varepsilon_-(\mathbf{q})|_{\mathrm{Im}\mathcal{M}=0} \equiv \varepsilon(\mathbf{q})$ and thus $\phi_+(\varepsilon(\mathbf{q}),\mathbf{q}) = \phi_+(-\varepsilon(\mathbf{q}),\mathbf{q})$.

In specifying the gluon propagator $D^{\alpha\beta}(p \equiv k - q)$, it is essential to take into account the long-range nature of the color magnetic interactions [16,18,20,32]. Once the effects of a normal medium are included in the random-phase approximation (RPA), these interactions are dynamically screened by the Landau damping of the virtual gluons involved [16,17]. Since the dominant contributions from $D(p)$ are peaked around $p = 0$, we focus our attention on $|p_0| \ll \mu/3$ and $|\mathbf{p}| \ll \mu/3$. The gluon propagator in the Landau gauge can be written within the RPA as

$$D_{\mu\nu}^{\alpha\beta}(p) = -\frac{\delta_{\alpha\beta}P_{\mu\nu}^T}{-p^2 + m_D^2\chi_T(p_0/|\mathbf{p}|)} - \frac{\delta_{\alpha\beta}P_{\mu\nu}^L}{-p^2 + m_D^2\chi_L(p_0/|\mathbf{p}|)} , \tag{70}$$

with the dimensionless transverse and longitudinal polarizations,

$$\chi_T(x) = \frac{x^2}{2} + \frac{x(1-x^2)}{4}\ln\left(\frac{x+1}{x-1}\right) , \tag{71}$$

$$\chi_L(x) = (1 - x^2)\left[1 - \frac{x}{2}\ln\left(\frac{x+1}{x-1}\right)\right] , \tag{72}$$

and transverse and longitudinal projection operators,

$$P_{ij}^T = \delta_{ij} - \frac{p_ip_j}{|\mathbf{p}|^2} , \quad P_{00}^T = P_{0i}^T = P_{i0}^T = 0 , \tag{73}$$

$$P_{\mu\nu}^L = \frac{p_\mu p_\nu}{p^2} - g_{\mu\nu} - P_{\mu\nu}^T , \tag{74}$$

where the Debye mass is given by

$$m_D = \left(\frac{g^2}{6\pi^2}\mathrm{Tr}\mathcal{M}^2 + \left(3 + \frac{N_f}{2}\right)\frac{g^2T^2}{3}\right)^{1/2} . \tag{75}$$

In Eq. (70), the term proportional to P^T describes the transverse or color magnetic sector, while that proportional to P^L describes the longitudinal or color electric sector. Landau damping provides an effective infrared cutoff in the magnetic sector, $\sim (\pi m_D^2|p_0|/4)^{1/3}$, for $|p_0| \ll |\mathbf{p}|$. This cutoff dominates the determination of the energy gap; the infrared cutoff given by a putative magnetic mass [20] makes negligible difference. In the presence of color pairing, certain of the magnetic gluons acquire a mass as a consequence of the Meissner effect [33,34], while the Debye mass of some of the electric gluons is reduced [34]. We ignore here these effects of the superconducting medium, although they possibly change the g dependence of the gap matrix in the weak coupling limit.

For the purpose of solving the gap equation (61) to leading logarithmic order in g, it is practical to approximate the polarized gluon propagator (70); we first set $-p^2 \approx |\mathbf{p}|^2$ and $\chi_L(x) \approx 1$, neglect the real part of $\chi_T(x)$, and replace $\mathrm{Im}\chi_T(x) = -\pi x/4$ for small $|x|$ by $-(\pi x/4)\theta(\sqrt{\pi}m_D/2 - |\mathbf{p}|)$. The effect of this latter replacement is to introduce a high momentum cutoff for the Landau-damped magnetic gluons. Then, we replace the gluon propagator by its real part, so that

$$D_{\mu\nu}^{\alpha\beta}(p) \simeq -\mathrm{Re}\left[\frac{\delta_{\alpha\beta}P_{\mu\nu}^T}{|\mathbf{p}|^2 - i\pi m_D^2 p_0\theta(\sqrt{\pi}m_D/2 - |\mathbf{p}|)/4|\mathbf{p}|}\right] - \frac{\delta_{\alpha\beta}P_{\mu\nu}^L}{|\mathbf{p}|^2 + m_D^2} . \tag{76}$$

This form is equivalent to that in Ref. [20]. In the gap equation (61), we may replace $|\mathbf{q}|$ by $|\mathbf{k}|$, and then replace $\hat{\mathbf{q}}$ by $\hat{\mathbf{k}}$ in the explicit Dirac structures $\mathrm{Tr}[\gamma^\mu\Lambda^-(\hat{\mathbf{q}})\gamma^\nu\Lambda^+(\hat{\mathbf{k}})]P_{\mu\nu}^{L,T}(k - q)$, so that both these terms become -2. We then find that the gap is determined by

$$\phi_+(\omega_\pm(\mathbf{k}), \mathbf{k}) = -\frac{g^2}{4} \sum_{\eta=\pm} \eta \int \frac{d^3q}{(2\pi)^3}$$

$$\times \left\{ \frac{|\mathbf{q}-\mathbf{k}|^2}{|\mathbf{q}-\mathbf{k}|^4 + (\pi m_D^2(\eta\varepsilon_\eta(\mathbf{q}) - \varepsilon_\pm(\mathbf{k}))/4|\mathbf{q}-\mathbf{k}|)^2\theta(\sqrt{\pi}m_D/2 - |\mathbf{q}-\mathbf{k}|)} \right.$$

$$\left. + \frac{1}{|\mathbf{q}-\mathbf{k}|^2 + m_D^2} \right\}$$

$$\times \frac{1}{4}(\lambda^\alpha)^T \phi_+(\omega_\eta(\mathbf{q}), \mathbf{q})\varepsilon_\eta^{-1}(\mathbf{q}) \tanh\left(\frac{\omega_\eta(\mathbf{q})}{2T}\right) \lambda^\alpha , \tag{77}$$

where θ is the angle between $\mathbf{k}$ and $\mathbf{q}$. The dominant contribution to the integral in Eq. (77) comes from colinear scattering between the quarks of momenta $\mathbf{k}$ and $\mathbf{q}$. Concentrating on the corresponding momentum region, $\cos\theta \simeq 1$ and $|\mathbf{k}| \simeq |\mathbf{q}|$, allows us to solve for the gap to subleading order in g by reducing Eq. (77) to

$$\phi_+(\omega_\pm(\mathbf{k}), \mathbf{k}) = \frac{g^2}{16\pi^2} \sum_{\eta=\pm} \eta \int_0^\infty d|\mathbf{q}| \left\{ \frac{1}{3}\ln\left[\frac{|\eta\varepsilon_\eta(\mathbf{q}) - \varepsilon_\pm(\mathbf{k})|}{M_T}\right] + \ln\left(\frac{M_T}{2|\mathbf{q}|}\right) + \ln\left(\frac{m_D}{2|\mathbf{q}|}\right) \right\}$$

$$\times \frac{1}{4}(\lambda^\alpha)^T \phi_+(\omega_\eta(\mathbf{q}), \mathbf{q})\varepsilon_\eta^{-1}(\mathbf{q}) \tanh\left(\frac{\omega_\eta(\mathbf{q})}{2T}\right) \lambda^\alpha , \tag{78}$$

with $M_T \equiv \sqrt{\pi}m_D/2$. Below, we shall consider the two optimal pairing states analyzed in Sec. II and estimate from Eqs. (69) and (78) the chemical potential differences between colors and the reduction in the pairing gaps induced by imposition of color neutrality.

A. Isoscalar, color-antitriplet channel

Cooper pairing in the isoscalar phase is described by a gap of the form (36), to within a constant phase factor and a special unitary transformation in flavor space. This gap matrix is generally accompanied by nonzero chemical potential differences between colors, as we found in the Ginzburg-Landau regime in Sec. II. We estimate here the chemical potential differences, using the gap equation (78) and the color neutrality conditions (3) and (4) with (69).

Note that the right sides of Eq. (69) as well as Eq. (78) with $\text{Im}\mathcal{M} = 0$ are integrals of an even function of $\varepsilon(\mathbf{q})$, since under $\varepsilon \to -\varepsilon$, $u_+^2 \leftrightarrow v_+^2$ and $f \leftrightarrow 1 - f$. It is thus convenient to diagonalize the matrix $\varepsilon^2(\mathbf{q})$ up to first order in the chemical potential differences, entering via $\tilde{\mathcal{M}}$. We first transform bases in color space from the original (RGB) basis to a new basis $(R'G'B')$ to diagonalize the matrix $\phi_+^\dagger(\mathbf{q})\phi_+(\mathbf{q})$ where $\phi_+(\mathbf{q}) \equiv \phi_+(\varepsilon(\mathbf{q}), \mathbf{q})$. When $\tilde{\mathcal{M}} = 0$, the unitary matrix U that carries out this diagonalization also diagonalizes $\varepsilon^2(\mathbf{q})$. As we shall see, U is determined solely by the direction of the vector $\mathbf{d}$, with components $d_a \equiv (\mathbf{d}_a)_s$, in complex color space and hence is independent of $\mathbf{q}$. In the isoscalar channel where the ij-components of $\phi_+^\dagger\phi_+$ obey $(\phi_+^\dagger\phi_+)_{(uu)} = (\phi_+^\dagger\phi_+)_{(dd)}$, $(\phi_+^\dagger\phi_+)_{(ss)} = 0$, and $(\phi_+^\dagger\phi_+)_{(ij)} = 0$ for $i \neq j$, it is sufficient to diagonalize the submatrix $(\phi_+^\dagger\phi_+)_{(uu)}$. The result reads

$$u^\dagger(\phi_+^\dagger\phi_+)_{(uu)}u = \begin{pmatrix} |\mathbf{d}|^2 & 0 & 0 \\ 0 & |\mathbf{d}|^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \tag{79}$$

where

$$u \equiv U_{uu} \equiv (\mathbf{u}_{R'}, \mathbf{u}_{G'}, \mathbf{u}_{B'}) , \tag{80}$$

with

$$\mathbf{u}_{R'} = \frac{\hat{\mathbf{d}}^* \times (\mathbf{e} \times \hat{\mathbf{d}})}{\sqrt{|\mathbf{e} \times \hat{\mathbf{d}}|^2}} , \quad \mathbf{u}_{G'} = \frac{\mathbf{e}^* \times \hat{\mathbf{d}}^*}{\sqrt{|\mathbf{e} \times \hat{\mathbf{d}}|^2}} , \quad \mathbf{u}_{B'} = \hat{\mathbf{d}} . \tag{81}$$

Here, $\hat{\mathbf{d}} \equiv \mathbf{d}/|\mathbf{d}|$, and $\mathbf{e}$ is an arbitrary complex unit vector that satisfies $\mathbf{e} \times \hat{\mathbf{d}} \neq 0$. Equation (79) describes the pairing state with the convention $d_{R'} = d_{G'} = 0$ and $d_{B'} \neq 0$. The fact that the the matrix $(\phi_+^\dagger\phi_+)_{(uu)}$ is effectively of rank two, corresponding to a reduction of the color symmetry from SU(3) to SU(2), implies that the condensate in the isoscalar, color-antitriplet channel is anisotropic in color space.

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The unitary matrix U can be taken to be block diagonal: $U_{(uu)} = U_{(dd)} = U_{(ss)} = u$ and $U_{(ij)} = 0$ for $i \neq j$. Then, multiplication of the gap equation (78) with $\mathrm{Im}\mathcal{M} = 0$ by U on the right and $U^\dagger$ on the left gives

$$u^\dagger [\phi_+(\mathbf{k})]_{(ud)} u \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \frac{g^2}{16\pi^2} \int_0^\infty d|\mathbf{q}| \left\{ \frac{1}{3} \ln \left[\frac{|u^\dagger \varepsilon_{(uu)}^2(\mathbf{k})u - u^\dagger \varepsilon_{(uu)}^2(\mathbf{q})u|}{M_T^2} \right] \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right.$$

$$\left. + \frac{1}{3} \ln \left[\frac{|u^\dagger \varepsilon_{(ss)}^2(\mathbf{k})u - u^\dagger \varepsilon_{(ss)}^2(\mathbf{q})u|}{M_T^2} \right] \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \ln \left(\frac{M_T^2}{4|\mathbf{q}|^2} \right) + \ln \left(\frac{m_D^2}{4|\mathbf{q}|^2} \right) \right\}$$

$$\times \frac{1}{4} u^\dagger (\lambda^\alpha)^T u u^\dagger [\phi_+(\mathbf{q})]_{(ud)} u u^\dagger \varepsilon_{(uu)}^{-1}(\mathbf{q}) \tanh \left(\frac{\varepsilon_{(uu)}(\mathbf{q})}{2T} \right) u u^\dagger \lambda^\alpha u \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \tag{82}$$

where $\varepsilon_{(ii)}$ is the ii-component of ε, and we have used $\varepsilon_{(uu)} = \varepsilon_{(dd)}$ and $\phi_{+(du)} = -\phi_{+(ud)}$. The Gell-Mann matrix terms can be transformed, via the Fierz identity $(\lambda^\alpha)_{ab}(\lambda^\alpha)_{cd} = -(2/3)\delta_{ab}\delta_{cd} + 2\delta_{ad}\delta_{bc}$, to

$$\sum_{abcd} u_{ba'}^*(\lambda^\alpha)_{ab} u_{ab'} u_{cc'}^*(\lambda^\alpha)_{cd} u_{dd'} = -\frac{2}{3} \delta_{a'b'}\delta_{c'd'} + 2(\mathbf{u}_{b'} \cdot \mathbf{u}_{d'})(\mathbf{u}_{a'} \cdot \mathbf{u}_{c'})^* . \tag{83}$$

Before taking into account the effects of nonzero $\tilde{\mathcal{M}}$, it is instructive to understand the structure of the gap equation with $\tilde{\mathcal{M}} = 0$. We first choose the complex unit vector $\mathbf{e}$ in such a way as to simplify the matrix $u^\dagger \phi_{+(ud)} u$, and then write down a reduced gap equation. Because of the invariance of the grand canonical Hamiltonian under color rotations in the absence of $\tilde{\mathcal{M}}$, it is sufficient to treat the states in which $\mathbf{d} \times \mathbf{d}^* = 0$. These states are related to the states in which $\mathbf{d} \times \mathbf{d}^* \neq 0$ by a gauge transformation, $\mathbf{d} \to e^{-2i\varphi}\mathbf{d}$, and a special unitary transformation, $\mathbf{d} \to U_c \mathbf{d}$.

For the states satisfying $\mathbf{d} \times \mathbf{d}^* = 0$, we can write $\mathbf{d} = |\mathbf{d}|\tilde{\mathbf{d}} \exp(i\varphi_0)$, where $\tilde{\mathbf{d}}$ is the real unit vector proportional to $\mathbf{d}$, and φ_0 is the phase common to all the elements of $\mathbf{d}$. Let us take $\mathbf{e}$ to be a real unit vector that satisfies $\mathbf{e} \times \mathbf{d} \neq 0$. We then obtain

$$u^\dagger \phi_{+(ud)} u = \begin{pmatrix} 0 & -|\mathbf{d}| & 0 \\ |\mathbf{d}| \exp(2i\varphi_0) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \tag{84}$$

as well as the relation, $\mathbf{u}_{a'} \cdot \mathbf{u}_{b'} = \delta_{a'b'}[\delta_{a'R'} + \delta_{a'G'}\exp(-2i\varphi_0) + \delta_{a'B'}\exp(2i\varphi_0)]$. The matrix equation (82) thus reduces to the single equation,

$$|\mathbf{d}(\mathbf{k})| = -\frac{g^2}{24\pi^2} \int_0^\infty d|\mathbf{q}| \left\{ \frac{1}{3} \ln \left(\frac{|E^2(\mathbf{q}) - E^2(\mathbf{k})|}{M_T^2} \right) + \ln \left(\frac{M_T^2}{4|\mathbf{q}|^2} \right) + \ln \left(\frac{m_D^2}{4|\mathbf{q}|^2} \right) \right\}$$

$$\times |\mathbf{d}(\mathbf{q})| E^{-1}(\mathbf{q}) \tanh \left(\frac{E(\mathbf{q})}{2T} \right) , \tag{85}$$

where

$$E(\mathbf{q}) = \left[\left(|\mathbf{q}| - \frac{\mu}{3} \right)^2 + |\mathbf{d}(\mathbf{q})|^2 \right]^{1/2} \tag{86}$$

is the $R'R'$ and $G'G'$ component of the diagonal matrix $\left([u^\dagger \varepsilon_{(uu)}^2(\mathbf{q})u]_{\tilde{\mathcal{M}}=0} \right)^{1/2}$. Note that Eq. (85) with $d_R = d_G = 0$ is equivalent to the gap equation usually analyzed for the isoscalar color-antitriplet channel (see, e.g., Refs. [20,32]). In the weak coupling limit of interest here, where a tiny energy gap arises in the momentum region immediately close to the Fermi surface, we can replace the $E^2(\mathbf{x})$ ($\mathbf{x} = \mathbf{k}, \mathbf{q}$) in the logarithmic term of Eq. (85) with the $B'B'$-component $(|\mathbf{x}| - \mu/3)^2$ of the diagonal matrix. We then assume in Eq. (85) that $|\mathbf{q}| \approx \mu/3$ in the last two logarithms and shift the integration variable from $|\mathbf{q}|$ to $|\mathbf{q}| - \mu/3$. The resulting equation reads

$$|\mathbf{d}(\mathbf{k})| = \frac{g^2}{18\pi^2} \int_0^\delta \frac{d(|\mathbf{q}| - \mu/3)}{E(\mathbf{q})} \tanh \left(\frac{E(\mathbf{q})}{2T} \right) \frac{1}{2} \ln \left[\frac{(b\mu/3)^2}{|(|\mathbf{q}| - \mu/3)^2 - (|\mathbf{k}| - \mu/3)^2|} \right] |\mathbf{d}(\mathbf{q})| , \tag{87}$$

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with

$$b \equiv 256\pi^4 \left(\frac{2}{3g^2} \right)^{5/2} ; \tag{88}$$

the cutoff δ, obeying $|\mathbf{d}(|\mathbf{q}| = \mu/3)| \ll \delta \ll m_D$, is chosen so that $|\mathbf{d}(|\mathbf{q}| > \delta)|$ is vanishingly small. Here, we have used $m_D^2 = g^2\mu^2/6\pi^2 + \mathcal{O}(g^2 T^2)$.

Let us now consider a color-singlet system and ask, in particular, whether or not the degeneracy of the order-parameter sets is removed in the weak coupling limit by the chemical potential differences between colors stemming from color neutrality (3) and (4). As a first step, we write the solution to the gap equation (87), following a line of argument of Pisarski and Rischke [20], and express the chemical potential deviations $\tilde{\mu}_{ab}$ in terms of the obtained gap. We then examine how such deviations in turn affect the gap itself. As we shall see, the degeneracy is not removed up to subleading order in g.

We now implement a process developed by Pisarski and Rischke [20] to convert the integral equation (87) into an equivalent differential equation [Eq. (92) below] via the approximation,

$$\frac{1}{2} \ln \left[\frac{(b\mu/3)^2}{|(|\mathbf{q}| - \mu/3)^2 - (|\mathbf{k}| - \mu/3)^2|} \right] \rightarrow \theta(||\mathbf{q}| - \mu/3| - ||\mathbf{k}| - \mu/3|) \ln \left(\frac{b\mu/3}{||\mathbf{q}| - \mu/3|} \right)$$
$$+ \theta(||\mathbf{k}| - \mu/3| - ||\mathbf{q}| - \mu/3|) \ln \left(\frac{b\mu/3}{||\mathbf{k}| - \mu/3|} \right) . \tag{89}$$

With this approximation and the change of variables,

$$x \equiv \ln \left[\frac{2b\mu/3}{||\mathbf{k}| - \mu/3| + E(\mathbf{k})} \right] , \quad y \equiv \ln \left[\frac{2b\mu/3}{||\mathbf{q}| - \mu/3| + E(\mathbf{q})} \right] , \quad x_F \equiv \ln \left[\frac{2b\mu/3}{E(|\mathbf{k}| = \mu/3)} \right] , \tag{90}$$

we obtain

$$|\mathbf{d}(x)| = \bar{g}^2 \left\{ x \int_x^{x_F} dy \tanh \left(\frac{E(y)}{2T} \right) |\mathbf{d}(y)| + \int_{\ln(b\mu/3\delta)}^x dy\, y \tanh \left(\frac{E(y)}{2T} \right) |\mathbf{d}(y)| \right\} , \tag{91}$$

where $\bar{g} \equiv g/3\sqrt{2}\pi$, and $\ln(b\mu/3||\mathbf{k}| - \mu/3|) \simeq x$ and $\ln(b\mu/3||\mathbf{q}| - \mu/3|) \simeq y$ have been used. Differentiation of Eq. (91) with respect to x leads finally to

$$\frac{d^2 |\mathbf{d}(x)|}{dx^2} = -\bar{g}^2 \tanh \left(\frac{E(x)}{2T} \right) |\mathbf{d}(x)| . \tag{92}$$

We summarize the results of Eq. (92), as obtained by Pisarski and Rischke [20]. At $T = 0$, the magnitude of the gap is given by

$$|\mathbf{d}(x)| = \frac{2}{3} b\mu e^{-\pi/2\bar{g}} \sin(\bar{g}x) . \tag{93}$$

It was pointed out that the BCS-like exponential term and the sinusoidal x dependence arise from nearly static magnetic gluons that undergo Landau damping and mediate the long-range part of the magnetic interactions, and that both the higher frequency magnetic gluons, which are little affected by Landau damping, and Debye-screened electric gluons play a dominant role in determining the pre-exponential factor.

The overall coefficient of the μ/g^5 in the prefactor is usually considered to be correct up to a factor of order unity, for the reason that radiative corrections such as quasiparticle wave function renormalization [18] and vertex corrections [32] do not modify the asymptotic form of the gap. However, it is still uncertain the extent to which this result is exact since contributions of the Meissner effect and Debye screening in the superconducting medium remain to be clarified [34].

Expression (93) indicates that the gap takes on a peak value $|\mathbf{d}(x_F)| = (2b\mu/3)e^{-\pi/2\bar{g}}$ at $|\mathbf{k}| = \mu/3$, smoothly reaches half its peak value at $||\mathbf{k}| - \mu/3| \sim |\mathbf{d}(x_F)|e^{\pi/3\bar{g}}$, and becomes of order $(g \ln g)|\mathbf{d}(x_F)|$ at $||\mathbf{k}| - \mu/3| \sim m_D$. The critical temperature T_c is given by

$$T_c = \frac{e^\gamma}{\pi} |\mathbf{d}(x_F, T = 0)| , \tag{94}$$

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where $\gamma = 0.5772\ldots$ is the Euler constant. It was stressed by Pisarski and Rischke [20] that expression (94) is of the usual BCS form, a feature stemming from the fact that in the weak coupling limit [35], $|\mathbf{d}(x,T)| \simeq |\mathbf{d}(x,T = 0)|(|\mathbf{d}(x_F,T)|/|\mathbf{d}(x_F,T = 0)|)$. This expression, except for a factor of order unity due to quasiparticle wave function renormalization, was reproduced by Brown, Liu, and Ren [36] using finite temperature diagrammatic perturbation theory in the normal phase. Since $\tilde{\mathcal{M}}$ vanishes at $T = T_c$, the critical temperature itself is unchanged by the color neutrality constraint.

We turn to the calculations of the chemical potential deviations $\tilde{\mu}_{ab}$ as functions of $|\mathbf{d}|$ given by Eq. (93); we consider only zero temperature for simplicity. Up to lowest order in $\tilde{\mathcal{M}}/\mu$, the difference between n_{ab} given by Eq. (69) and the corresponding normal-phase value $\delta_{ab} n_N \equiv \delta_{ab} 2(N_f/3)N(\mu/3)\mu/3$ can be written as

$$ n_{ab} - n_N \delta_{ab} = 2N_f N(\mu/3)\tilde{\mu}_{ab} + 2 \sum_{a'=R',G'} u_{aa'} u_{ba'}^* \delta n \ , \tag{95}$$

where

$$ \delta n \equiv \frac{1}{V} \left\{ \sum_{|\mathbf{q}|<\mu/3} \left[-1 - \frac{|\mathbf{q}| - \mu/3}{E(\mathbf{q})} \right] + \sum_{|\mathbf{q}|>\mu/3} \left[1 - \frac{|\mathbf{q}| - \mu/3}{E(\mathbf{q})} \right] \right\} \tag{96}$$

is the number density excess due to pairing [37], and V is the system volume. This excess can be estimated from Eqs. (86) and (93) as

$$ \delta n \simeq \frac{3\pi}{2\bar{g}\mu} N(\mu/3)|\mathbf{d}(x_F)|^2 \ . \tag{97}$$

Using these expressions in the color neutrality conditions (3) and (4), we obtain, with the help of Eqs. (7) and (81),

$$ \tilde{\mu}_{ab} = (3\hat{d}_a \hat{d}_b^* - \delta_{ab}) \frac{\delta n}{9N(\mu/3)} \ . \tag{98}$$

The chemical potential deviations $\tilde{\mu}_{ab}$ are thus of order $|\mathbf{d}(x_F)|^2/\bar{g}\mu$.

Up to subleading order in g, such chemical potential deviations do not modify any of the factors in Eq. (93) for the magnitude of the gap $|\mathbf{d}|$. This is because $||\mathbf{q}| - \mu/3||\tilde{\mu}_{ab}| \ll |\mathbf{d}(x_F)|^2$ in the momentum region, $||\mathbf{q}| - \mu/3| \lesssim \delta$, where the gap is appreciable. We remark that this robustness of the exponential term and the leading pre-exponential factor in Eq. (93) is supported by the renormalization-group analysis developed by Son [18]. For example, let us consider the case in which $d_R = d_G = 0$ and $d_B \neq 0$. In this case, Cooper pairing occurs between the quarks of color R and G with momenta $\pm\mathbf{k}$ close to the Fermi surface having the Fermi momentum $k_F = \mu/3 + \tilde{\mu}_{RR} = \mu/3 + \tilde{\mu}_{GG}$. In the renormalization-group analysis, the scattering amplitude between these quarks is characterized by the parameter $t = -\ln||\mathbf{k}| - k_F|$. The scattering amplitude becomes singular, as Gorkov originally showed, at the onset of pairing. The value of t at which this singularity occurs – the Landau pole t_L – tells us the scale of the energy gap according to $|\mathbf{d}(|\mathbf{k}| = k_F)| \sim m_D e^{-t_L}$. Up to subleading order in g, non-zero $\tilde{\mu}_{RR}$ has no effect on t_L, which behaves as $t_L = \pi/2\bar{g} + 6\ln g + \cdots$.

In the weak coupling limit, up to subleading order in g, we thus find that the degenerate set of $\mathbf{d}$'s obtained with $\tilde{\mathcal{M}} = 0$ persists in the color-singlet system studied here, a result consistent with that obtained from the general Ginzburg-Landau theory in Sec. II. Nevertheless, it is instructive to examine the leading contribution of $\tilde{\mu}_{ab}$ to the zero-temperature gap, in view of the fact that a strongly coupled quark system may develop a considerable energy gap (see, e.g., Ref. [13]).

The properties of this contribution can be roughly understood by finding the eigenvalues of $\varepsilon_{(uu)}^2(\mathbf{q})$, $= \varepsilon_{(dd)}^2(\mathbf{q})$, the square of the quasiparticle energy given by Eq. (60). This analysis is only applicable to the case in which $\mathbf{d} \times \mathbf{d}^* = 0$ and hence $\mathrm{Im}M = 0$. (If $\mathrm{Im}M \neq 0$, one must go back to the original gap equation (78) dependent on the matrices ε_η and ω_η, which are not diagonalized by the same unitary transformation.) Up to $\mathcal{O}(\tilde{\mathcal{M}})$, these eigenvalues are given by

$$ E_{R'}^2(\mathbf{q}) = \left(|\mathbf{q}| - \frac{\mu}{3} - \delta\mu_{R'}\right)^2 + |\mathbf{d}(\mathbf{q})|^2 \ , $$

$$ E_{G'}^2(\mathbf{q}) = \left(|\mathbf{q}| - \frac{\mu}{3} - \delta\mu_{G'}\right)^2 + |\mathbf{d}(\mathbf{q})|^2 \ , $$

$$ E_{B'}^2(\mathbf{q}) = \left(|\mathbf{q}| - \frac{\mu}{3} - \delta\mu_{B'}\right)^2 \ , \tag{99}$$

where

$$\delta\mu = \delta\mu_{B'} = -2\delta\mu_{R'} = -2\delta\mu_{G'} \equiv \frac{\pi|\mathbf{d}(x_F)|^2}{3\bar{g}\mu} \tag{100}$$

is the chemical potential shift from $\mu/3$. The order of the eigenvalues corresponds to that in Eq. (79). Note that the chemical potential shift $\delta\mu$, stemming from the anisotropy of the condensate in color space, is independent of its color orientation. This reflects the fact that the properties of a color-singlet system do not depend on the choice of color axes. It is also important to note that $\delta\mu$ is positive definite. This feature arises due to the property discussed in note [37]. Not only does positive $\delta\mu$ enlarge the Fermi surface of the gapless quarks of color B', but it also decreases the Fermi energy of the quarks of colors R' or G', leading to overall suppression of the energy gap.

In order to calculate in detail the $\mathcal{O}(\tilde{\mu}_{ab})$ correction to the magnitude of the gap in the case in which $\mathbf{d} \times \mathbf{d}^* = 0$, we note that U diagonalizes $\mathcal{M}$ and hence $\varepsilon^2(\mathbf{q})$. This is evident from the relation,

$$\mathcal{M}_{abij} = \delta_{ij}\left\{\left[\frac{\mu}{3} + \frac{\pi|\mathbf{d}(x_F)|^2}{3\bar{g}\mu}\right]\delta_{ab} - \frac{\pi[(\phi_+^\dagger\phi_+)_{(uu)}]_{|\mathbf{k}|=\mu/3}}{2\bar{g}\mu}\right\} , \tag{101}$$

obtained from Eq. (98). It is thus straightforward to show that the gap equation (82) reduces to Eq. (85) in which the quasiparticle energy $E(\mathbf{q})$ is replaced by $E_{R'}(\mathbf{q})$, $=E_{G'}(\mathbf{q})$. The solution to this equation reads

$$|\mathbf{d}(x')| = \frac{2C_1 b\mu}{3}\exp\left(-\frac{\pi}{2\bar{g}}\right)\sin(\bar{g}x') ,$$

$$x' \equiv \ln\left[\frac{2C_1 b\mu/3}{||\mathbf{k}| - \mu/3 + \delta\mu/2| + E_{R'}(\mathbf{k})}\right] , \tag{102}$$

where the correction factor is

$$C_1 \simeq 1 - 9\frac{\delta\mu}{\mu} , \tag{103}$$

and we have used $m_D^2 = g^2\mu^2/6\pi^2 + \mathcal{O}(g^2\delta\mu^2)$.

Note that C_1 is less than unity. As a consequence the anisotropy of the condensate in a color-singlet system acts to reduce the gap size.

This reduction, together with the reduction in the densities of states at the Fermi surfaces for colors R' and G' due to $\delta\mu_{R',G'}$, leads to a decrease in the condensation energy of the system. These features arise despite the fact that there are no Fermi momentum differences between the two colors associated with the pairing. This situation is in contrast to the case of ferromagnetic superconductors [38] and to the case of color-flavor-locked quark superfluids with nonzero strange quark mass [39], where Fermi momentum mismatches between the electron spins and between the quark flavors involved in the pairing, respectively, play a role in reducing the magnitude of the gap.

B. Color-flavor locking

We proceed to examine color-flavor locking, described by the order parameter (39) to within a constant phase factor and a special unitary transformation in flavor space. We first note the relation $\tilde{\mu}_{ab} \propto \sum_i[(\phi_+^\dagger\phi_+)_{|\mathbf{k}|=\mu/3}]_{abii} +$ const. $\times \delta_{ab}$, which can be derived up to first order in $\phi_+^\dagger\phi_+$ from the color-singlet conditions (3) and (4) as well as Eq. (69). Here, the constant affixed to δ_{ab} ensures Eq. (7). Substituting Eq. (39) into this relation we find $\sum_i[(\phi_+^\dagger\phi_+)_{|\mathbf{k}|=\mu/3}]_{abii} = 4\delta_{ab}|\kappa_A|^2$, and thus obtain $\tilde{\mathcal{M}} = 0$. This result, coming from the isotropy of the condensate in color space, is consistent with that from the general Ginzburg-Landau analysis in Sec. II. We can thus derive the behavior of the energy gap at $T = 0$ and near T_c from the gap equation (78) with $\tilde{\mathcal{M}} = 0$. Substitution of ansatz (39) and $\tilde{\mathcal{M}} = 0$ into Eq. (78) leads to an equation for κ_A,

$$\kappa_A(\mathbf{k}) = \frac{g^2}{18\pi^2}\int_0^\delta d(|\mathbf{q}| - \mu/3)\frac{1}{2}\ln\left[\frac{(b\mu/3)^2}{|(|\mathbf{q}| - \mu/3)^2 - (|\mathbf{k}| - \mu/3)^2|}\right]$$

$$\times\left\{-\frac{2}{3}\left[\frac{-\kappa_A(\mathbf{q})}{E_{(8)}(\mathbf{q})}\right]\tanh\left(\frac{E_{(8)}(\mathbf{q})}{2T}\right)\right.$$

$$\left.+\frac{1}{6}\left[\frac{2\kappa_A(\mathbf{q})}{E_{(1)}(\mathbf{q})}\right]\tanh\left(\frac{E_{(1)}(\mathbf{q})}{2T}\right)\right\} , \tag{104}$$

with

$$E_{(8)}(\mathbf{q}) = \left((|\mathbf{q}| - \mu/3)^2 + |\kappa_A(\mathbf{q})|^2\right)^{1/2} ,$$
$$E_{(1)}(\mathbf{q}) = \left((|\mathbf{q}| - \mu/3)^2 + 4|\kappa_A(\mathbf{q})|^2\right)^{1/2} . \tag{105}$$

Here, we have used the color structure,

$$\sum_{bc} \frac{1}{4}(\lambda^\alpha)_{ba}(\delta_{bi}\delta_{cj} - \delta_{bj}\delta_{ci})(\lambda^\alpha)_{cd} = -\frac{2}{3}(\delta_{ai}\delta_{dj} - \delta_{aj}\delta_{di}) , \tag{106}$$

and the diagonalized form of the quasiparticle energy squared,

$$U_{\mathrm{CFL}}^T \varepsilon^2(\mathbf{q}) U_{\mathrm{CFL}} = \delta_{a'b'}\delta_{i'j'}E_{(8)}^2(\mathbf{q}) + \delta_{a'B'}\delta_{b'B'}\delta_{i's'}\delta_{j's'}[E_{(1)}^2(\mathbf{q}) - E_{(8)}^2(\mathbf{q})] , \tag{107}$$

with

$$U_{\mathrm{CFL}} = \frac{1}{\sqrt{6}}\begin{pmatrix} -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2} \\ 0 & \sqrt{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{6} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{6} & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -\sqrt{3} & 0 & 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 & 0 & 0 & \sqrt{6} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{6} & 0 \\ 1 & 0 & 0 & 0 & \sqrt{3} & 0 & 0 & 0 & \sqrt{2} \end{pmatrix} ; \tag{108}$$

here the bases are taken to be $(Ru, Gu, Bu, Rd, Gd, Bd, Rs, Gs, Bs)$ in the original color-flavor space and $(R'u', G'u', B'u', R'd', G'd', B'd', R's', G's', B's')$ in the transformed color-flavor space.

At $T = 0$, the gap equation (104) is equivalent to that obtained by Schäfer [11]. The analogy between Eq. (104) and Eq. (87) allows us to write κ_A in the form (93) valid up to subleading order in g [11]:

$$\kappa_A(x_{(8)}) = 2^{-1/3}\exp(i\varphi_A)\frac{2b\mu}{3}\exp\left(-\frac{\pi}{2\bar{g}}\right)\sin(\bar{g}x_{(8)}) , \tag{109}$$

where

$$x_{(8)} \equiv \ln\left(\frac{2^{2/3}b\mu/3}{||\mathbf{k}| - \mu/3| + E_{(8)}(\mathbf{k})}\right) , \tag{110}$$

and φ_A is the phase of κ_A.

At finite temperatures, the assumption that $\kappa_A(\mathbf{k}, T) \simeq \kappa_A(\mathbf{k}, T = 0)(\kappa_A(|\mathbf{k}| = \mu/3, T)/\kappa_A(|\mathbf{k}| = \mu/3, T = 0))$ holds as in the isoscalar, color-antitriplet channel. At $T = T_c$, where $E_{(8)}(\mathbf{q}) = E_{(1)}(\mathbf{q})$, the gap equation (104) has the same structure as Eq. (87) obtained for the isoscalar, color-antitriplet channel. Thus, the critical temperature T_c equals the result, Eq. (94), in the isoscalar channel. Such equality, persisting in the strong coupling regime (see Sec. II), stems from the fact that both these types of pairing are induced by the same instability of the normal phase.

IV. GINZBURG-LANDAU REGION AT HIGH DENSITIES

We are now in a position to calculate, in the weak coupling limit, the Ginzburg-Landau free energy, (15), the difference between the superfluid and normal phases near the transition temperature T_c. As earlier, we ignore the normal state Hartree-Fock terms in the Schwinger-Dyson equation (49), and identify the thermodynamic potential in the normal phase with that of an ultrarelativistic, noninteracting Fermi gas of Fermi energy $\mu/3$. We thus obtain [21]

$$\Delta\Omega = \Delta\Omega_{\mathrm{ideal}} + \Omega_{\mathrm{loop}} + \Omega_{\mathrm{res}} , \tag{111}$$

where

$$\Delta\Omega_{\mathrm{ideal}} = -\frac{1}{12\pi^2}\left[\mathrm{Tr}\mathcal{M}^4 - 3N_f\left(\frac{\mu}{3}\right)^4\right] \tag{112}$$

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is the difference in the ideal-gas contribution between the superfluid and normal phases;

$$\Omega_{\text{loop}} = -\frac{T}{2} \sum_{n \text{ odd}} \int \frac{d^3q}{(2\pi)^3} \text{Tr}[-G(q)\Sigma(q) + \ln G_0^{-1}(q)G(q)] \tag{113}$$

is the contribution, again up to $\mathcal{O}(g^2)$, of quark loops in the superfluid vacuum; and

$$\Omega_{\text{res}} = -\left\{ \frac{T}{4} \sum_{n \text{ odd}} \int \frac{d^3q}{(2\pi)^3} \text{Tr}[G(q)\Sigma(q)] \right\}_{T=T_c, \tilde{\mathcal{M}}=0}, \tag{114}$$

resulting from the mean-field approximation [19] adopted in writing Eq. (49), guarantees that the energy gap Δ vanishes at $T = T_c$.

We now expand the thermodynamic potential difference (111) with respect to $\Delta(\mathbf{q}) \equiv \Delta(q_0 = |\mathbf{q}| - \mu/3, \mathbf{q})$ and $\tilde{\mu}_{ab}$. In calculating Ω_{loop} up to $\mathcal{O}(\Delta^4)$ and $\mathcal{O}(\Delta^2\tilde{\mathcal{M}})$, we use the quark propagator G given by Eq. (50), the proper self energy Σ given by Eq. (51), and the noninteracting quark propagator G_0 given by Eq. (52).

The explicit Dirac structure of G can be obtained from $G^{(11)}$, Eq. (63), $G^{(21)}$, Eq. (54),

$$G^{(12)}(q) = \left[\tilde{\Delta}(q)(\gamma q - \gamma^0 \mathcal{M}^T)^{-1} \Delta(q) - (\gamma q + \gamma^0 \mathcal{M}) \right]^{-1} \tilde{\Delta}(q)(\gamma q - \gamma^0 \mathcal{M}^T)^{-1}, \tag{115}$$

and

$$G^{(22)}(q) = \left[(\gamma q - \gamma^0 \mathcal{M}^T) - \Delta(q)(\gamma q + \gamma^0 \mathcal{M})^{-1} \tilde{\Delta}(q) \right]^{-1}. \tag{116}$$

Thus,

$$\text{Tr}[G(q)\Sigma(q)] = 2\text{Tr}\left\{ \frac{1}{\gamma q + \gamma^0 \mathcal{M}} \tilde{\Delta}(q) \frac{1}{\gamma q - \gamma^0 \mathcal{M}^T} \Delta(q) \right.$$
$$\left. + \left[\frac{1}{\gamma q + \gamma^0 \mathcal{M}} \tilde{\Delta}(q) \frac{1}{\gamma q - \gamma^0 \mathcal{M}^T} \Delta(q) \right]^2 + \cdots \right\}, \tag{117}$$

and

$$\text{Tr}[\ln G_0^{-1}(q)G(q)] = \text{Tr}\left\{ \frac{1}{\gamma q + \gamma^0 \mathcal{M}} \tilde{\Delta}(q) \frac{1}{\gamma q - \gamma^0 \mathcal{M}^T} \Delta(q) \right.$$
$$\left. + \frac{1}{2} \left[\frac{1}{\gamma q + \gamma^0 \mathcal{M}} \tilde{\Delta}(q) \frac{1}{\gamma q - \gamma^0 \mathcal{M}^T} \Delta(q) \right]^2 + \cdots \right\}. \tag{118}$$

Combining Eqs. (117) and (118) with (113) and performing the Matsubara frequency summation, we finally obtain

$$\Omega_{\text{loop}} = \Omega_{\text{loop}}^{(0)} + \Omega_{\text{loop}}^{(1)}, \tag{119}$$

where

$$\Omega_{\text{loop}}^{(0)} = -\frac{1}{2} \int \frac{d^3q}{(2\pi)^3} \tanh\left(\frac{|\mathbf{q}| - \mu/3}{2T} \right) \frac{1}{|\mathbf{q}| - \mu/3} \text{Tr}[\phi_+^\dagger(\mathbf{q})\phi_+(\mathbf{q})]$$
$$+ \frac{3}{8} \int \frac{d^3q}{(2\pi)^3} \left[\tanh\left(\frac{|\mathbf{q}| - \mu/3}{2T} \right) \frac{1}{(|\mathbf{q}| - \mu/3)^3} \right.$$
$$\left. - \cosh^{-2}\left(\frac{|\mathbf{q}| - \mu/3}{2T} \right) \frac{1}{2T(|\mathbf{q}| - \mu/3)^2} \right]$$
$$\times \text{Tr}[\phi_+^\dagger(\mathbf{q})\phi_+(\mathbf{q})\phi_+^\dagger(\mathbf{q})\phi_+(\mathbf{q})] \tag{120}$$

is the usual Ginzburg-Landau expansion obtained for $\tilde{\mathcal{M}} = 0$, and

$$\Omega_{\text{loop}}^{(1)} = -\frac{1}{2} \int \frac{d^3q}{(2\pi)^3} \left[\tanh\left(\frac{|\mathbf{q}| - \mu/3}{2T} \right) \frac{1}{(|\mathbf{q}| - \mu/3)^2} \right.$$
$$\left. - \cosh^{-2}\left(\frac{|\mathbf{q}| - \mu/3}{2T} \right) \frac{1}{2T(|\mathbf{q}| - \mu/3)} \right] \text{Tr}[\phi_+^\dagger(\mathbf{q})\phi_+(\mathbf{q})\tilde{\mathcal{M}}] \tag{121}$$

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is the $\mathcal{O}(\tilde{\mathcal{M}})$ correction to the first term on the right side of Eq. (120). We have again ignored the contributions of antiquark quasiparticles, and used the relation $\mathrm{Tr}(\phi_+^\dagger \phi_+ \tilde{\mathcal{M}}) = \mathrm{Tr}(\phi_+ \phi_+^\dagger \tilde{\mathcal{M}}^T)$, coming from $\phi_+^T(\mathbf{q}) = \phi_+(\mathbf{q})$ [see Eq. (10)].

We turn now to calculating $\Omega_{\mathrm{loop}}^{(0)}$ and $\Omega_{\mathrm{loop}}^{(1)}$ for the isoscalar, color-antitriplet channel considered in Secs. II and III, and derive the high-density Ginzburg-Landau form of $\Delta\Omega$ used in Sec. II. We remark in passing that the calculations of $\Omega_{\mathrm{loop}}^{(0)}$ for the color-flavor locking reproduce the values of $\bar{\alpha}$, β_1, and β_2 that will be derived below for the isoscalar channel.

To derive the coefficients of the Ginzburg-Landau free energy for isoscalar, color-antitriplet pairing, we first carry out the integrals over $\mathbf{q}$ in $\Omega_{\mathrm{loop}}^{(0)}$, Eq. (120), and $\Omega_{\mathrm{loop}}^{(1)}$, Eq. (121). Here we evaluate $\phi_+(\mathbf{q}, T)$, whose $\mathbf{q}$ and T dependences are effectively decoupled for $T \leq T_c$ (see note [35]), as

$$\phi_+(\mathbf{q}, T) \simeq \phi_+(|\mathbf{q}| = \mu/3, T)\sin(\bar{g}y)|_{T=0} , \tag{122}$$

where y is given by Eq. (90). We also use the approximation,

$$\tanh\left(\frac{|\mathbf{q}| - \mu/3}{2T}\right) \rightarrow \theta(\kappa|\mathbf{d}(x_F, T = 0)| - ||\mathbf{q}| - \mu/3|)\tanh\left(\frac{|\mathbf{q}| - \mu/3}{2T}\right)$$
$$+ \theta(||\mathbf{q}| - \mu/3| - \kappa|\mathbf{d}(x_F, T = 0)|)\frac{|\mathbf{q}| - \mu/3}{||\mathbf{q}| - \mu/3|} , \tag{123}$$

where κ is a positive number of order g^{-1}, and x_F is given by Eq. (90). With this choice, $\tanh((|\mathbf{q}| - \mu/3)/2T) \simeq (|\mathbf{q}| - \mu/3)/||\mathbf{q}| - \mu/3|$ for $||\mathbf{q}| - \mu/3| \gtrsim \kappa|\mathbf{d}(x_F, T = 0)|$, and $\sin(\bar{g}y)|_{T=0} \simeq 1$ for $||\mathbf{q}| - \mu/3| \lesssim \kappa|\mathbf{d}(x_F, T = 0)|$. We thus obtain

$$\Omega_{\mathrm{loop}}^{(0)} = -N(\mu/3)\ln\left[\frac{e^\gamma(2b\mu/3)^{1/2}|\mathbf{d}(x_F, T = 0)|^{1/2}}{\pi T}\right]\mathrm{Tr}(\phi_+^\dagger \phi_+)_F$$
$$+ \frac{3}{2}\frac{7\zeta(3)}{8(\pi T_c)^2}N(\mu/3)\mathrm{Tr}(\phi_+^\dagger \phi_+ \phi_+^\dagger \phi_+)_F , \tag{124}$$

and

$$\Omega_{\mathrm{loop}}^{(1)} = \frac{3}{\mu}\ln\left(\frac{3T_c}{\mu}\right)N(\mu/3)\mathrm{Tr}(\phi_+^\dagger \phi_+ \tilde{\mathcal{M}})_F \equiv \chi\mathrm{Tr}(\phi_+^\dagger \phi_+ \tilde{\mathcal{M}})_F , \tag{125}$$

where the coefficients of $\mathrm{Tr}(\cdots)_F$ include the leading contributions with respect to g, and the temperature has been set equal to the critical temperature Eq. (94), except in the coefficient of $\mathrm{Tr}(\phi_+^\dagger \phi_+)_F$. The term Ω_{res}, expressed as

$$\Omega_{\mathrm{res}} = N(\mu/3)\ln\left[\frac{e^\gamma(2b\mu/3)^{1/2}|\mathbf{d}(x_F, T = 0)|^{1/2}}{\pi T_c}\right]\mathrm{Tr}(\phi_+^\dagger \phi_+)_F$$
$$- \frac{7\zeta(3)}{8(\pi T_c)^2}N(\mu/3)\mathrm{Tr}(\phi_+^\dagger \phi_+ \phi_+^\dagger \phi_+)_F , \tag{126}$$

acts as a counterterm to the coefficient of $\mathrm{Tr}(\phi_+^\dagger \phi_+)_F$, leading to $\Delta\Omega = 0$ at $T = T_c$ [2]. Then the sum of $\Omega_{\mathrm{loop}}^{(0)}$ and Ω_{res} reduces to the usual Ginzburg-Landau part Ω_0,

$$\Omega_0 = N(\mu/3)\ln\left(\frac{T}{T_c}\right)\mathrm{Tr}(\phi_+^\dagger \phi_+)_F + \frac{1}{2}\frac{7\zeta(3)}{8(\pi T_c)^2}N(\mu/3)\mathrm{Tr}(\phi_+^\dagger \phi_+ \phi_+^\dagger \phi_+)_F$$
$$\equiv \alpha^+\mathrm{Tr}(\phi_+^\dagger \phi_+)_F + \beta_2^+\mathrm{Tr}[(\phi_+^\dagger \phi_+)^2]_F . \tag{127}$$

From Eq. (125) we derive the value of χ, (34). Equation (127) indicates that $\beta_1^+ = 0$, and implies the coefficients in Eqs. (30)-(32).

The coefficients in Eq. (127) agree with those obtained by Bailin and Love [2] for a BCS-type short-range pairing interaction and for $\tilde{\mathcal{M}} = 0$. The agreement with the term of fourth order in ϕ_+ arises because the main contribution to the corresponding integral in Eq. (120) comes from the momentum region $||\mathbf{q}| - \mu/3| \lesssim \kappa|\mathbf{d}(x_F, T = 0)|$, where the gap is almost flat. The agreement for the second order term in ϕ_+ is obvious. On the other hand, the coefficient

of $\text{Tr}(\phi_+^\dagger \phi_+ \tilde{\mathcal{M}})_F$ in $\Omega_{\text{loop}}^{(1)}$, as can be seen from Eq. (125), is different from $-(6/\mu)N(\mu/3)\ln(\Lambda/T_c)$ with the ultra-violet cutoff Λ as in a BCS superconductor, due to the behavior $\phi_+(\mathbf{q}, T) \propto \sin(\bar{g}y)|_{T=0}$ induced by the long-range dynamically screened magnetic interactions.

The difference in the ideal-gas free energy of the superfluid and normal phases, $\Delta\Omega_{\text{ideal}}$, Eq. (112), gives rise to a term proportional to $\text{Tr}\tilde{\mathcal{M}}^2$, with coefficient given by Eq. (33). The color neutrality conditions, (19) and (20), thus imply

$$\tilde{\mu}_{ab} = -\frac{1}{3\mu}\ln\left(\frac{3T_c}{\mu}\right)[3d_a(x_F, T)d_b^*(x_F, T) - |\mathbf{d}(x_F, T)|^2\delta_{ab}] , \tag{128}$$

in agreement with Eq. (44) with $(\mathbf{d}_a)_i = \delta_{is}d_a$. Comparing this expression to the $T = 0$ result Eq. (98), we find that $\tilde{\mu}_{ab}/|\mathbf{d}(x_F, T)|^2$ is identical at both $T = 0$ and $T \simeq T_c$. Expression (128) can also be derived from the same analysis that yields Eq. (98) if we replace, in Eq. (95), δn as given by Eq. (96) with

$$\delta n = V^{-1}\sum_{\mathbf{q}}\left\{\tanh\left(\frac{|\mathbf{q}| - \mu/3}{2T}\right) - \frac{|\mathbf{q}| - \mu/3}{E(\mathbf{q})}\tanh\left(\frac{E(\mathbf{q})}{2T}\right)\right\} . \tag{129}$$

The equality in $\tilde{\mu}_{ab}/|\mathbf{d}(x_F, T)|^2$ in the $T = 0$ and $T \simeq T_c$ cases arises from the fact that both have nearly the same momentum dependence of the gap.

V. CONCLUSION

In this paper we have laid out the Ginzburg-Landau structure of superconducting quark matter. Even in the homogeneous case considered here, many questions remain. First, electrical charge neutrality, in addition to color neutrality, should be duly taken into account. The system considered here is composed of three flavor massless quarks in flavor equilibrium, as characterized by Eq. (1). Such matter, when normal, is electrically neutral in itself. Below T_c, however, it has nonzero net electric charge, unless the order parameter is isotropic in flavor space. This isotropy is retained by the color-flavor locked phase; however, in the isoscalar channel, strange quarks remain gapless, leading to a deficit of the total number of strange quarks relative to the total baryon number, and hence to positive net charge. This charge, in neutron star matter, would be neutralized by charged leptons.

It is straightforward to extend the formalism obtained in the absence of leptons to the situation of electrical charge neutrality. When the system is in overall beta equilibrium, the quark chemical potentials, for each color a, obey $\mu_{au} + \mu_e = \mu_{ad} = \mu_{as}$, where μ_e is the electron chemical potential. Then the quark chemical potential, μ_a, for color a, defined by

$$\mu_a \equiv \mu_{ai} + q_i\mu_e , \tag{130}$$

is the same for all flavors i; here, q_i is the electric charge of the quark of flavor i. This relation replaces condition (1). As a consequence of electrical charge neutrality, Eq. (5) remains valid. We keep the same definition (6) of the $\tilde{\mu}_{aa}$. The matrix $\tilde{\mathcal{M}}$, given by Eq. (21), can now be rewritten as a traceless matrix

$$\tilde{\mathcal{M}}_{abij} = \delta_{ij}(\tilde{\mu}_{ab} - q_i\mu_e\delta_{ab}) . \tag{131}$$

Near T_c, $\Delta\Omega$ is still given by Eqs. (24) and (27), since the lepton pressure, which is of order μ_e^4 [$\sim \mathcal{O}(\tilde{\mu}_{ab}^4)$], is negligible compared with Ω_{CN} which is of order $\mu^2\tilde{\mu}_{ab}^2$. (Up to leading order in $\mathbf{d}_a$, we obtain $\mu_e = -(\chi/2\sigma)\sum_{ai}q_i[(\mathbf{d}_a^*)_i(\mathbf{d}_a)_i]_F + \mathcal{O}(\mu_e^3/\mu^2)$ from the electric neutrality condition, $\partial\Delta\Omega/\partial\mu_e = 0$, and Eq. (44) from the color neutrality conditions (19) and (20). Thus, $\mu_e \sim \mathcal{O}(\tilde{\mu}_{ab})$.) Since electricity does not distinguish between colors, the presence of μ_e in the isoscalar channel $[(\mathbf{d}_a)_i = \delta_{is}d_a]$ leads to a Fermi momentum mismatch between u and d quarks, and thus results in a uniform suppression in the magnitudes of the gap and condensation energy; in Eqs. (47) and (48), β_1 is replaced by $\beta_1 + \Gamma/2$. Accordingly, in the phase diagram illustrated in Fig. 2, the isoscalar phase is shifted by $\Gamma/4$ in the β_1 direction and $-3\Gamma/4$ in the β_2 direction. Note that the general flavor-antitriplet states belonging to this phase can no longer be reduced to the isoscalar channel by flavor rotations since the value of μ_e and the gap suppression depend on the the electric charges of quarks involved in Cooper pairing.

A second important problem is how the strange quark mass affects the quark superfluidity. A non-zero strange quark mass not only breaks invariance of the grand-canonical Hamiltonian with respect to rotations in flavor space, but it also necessitates the presence of a neutralizing gas of leptons even in the normal phase. The resulting m_s dependence of the stability of color-flavor locking over the isoscalar state was considered by several authors (see, e.g.,

Refs. [39,40]), who found that in contrast to the case of the isoscalar channel, color-flavor locking is destabilized by the Fermi momentum mismatch between s quarks and u, d quarks produced by a nonzero value of m_s.

It is instructive to consider the effect of m_s in terms of the general Ginzburg-Landau approach constructed here. In the highly relativistic regime ($m_s \ll \mu/3$), in addition to the terms in the massless limit, $\Delta\Omega$ contains a term proportional to $\mathrm{Tr}(\phi_+^\dagger \phi_+ M^2)_F$, where $M_{abij} = \delta_{ab}\delta_{ij}m_i$ with $m_u = m_d = 0$. This term tends to suppress the pairing gap just like the term proportional to $\mathrm{Tr}(\phi_+^\dagger \phi_+ \tilde{\mathcal{M}})_F$ coming from color neutrality. Note that because of color-flavor locking, the resulting anisotropy of the condensate in flavor space acts to fix its color orientation. The transition temperature T_c is also reduced in the color-flavor locked state, leading to a T_c in the color-flavor locked state smaller than T_c in the isoscalar channel, which is independent of m_s. The equilibrium phase diagram for the superfluid transition, in the μ versus T plane, is sensitive not only to the effects of the color neutrality and the strong coupling, as stressed in the present work, but also to effects of electric neutrality and finite strange quark mass.

ACKNOWLEDGMENTS

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